Chapter 8

Stress Energy Tensor

8.1 Conservation of energy in classical mechanics

Let us consider a classical system with 1 degree of freedom, described by the generalized coordinate q. Let the system admit a Lagrangian formulation, and let $L\left(q, \frac{dq}{dt}, t\right)$ be the Lagrangian of the system. In terms of the Lagrangian the dynamics of the system is described by the Euler-Lagrange equations, i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

We now make the additional hypothesis that the Lagrangian *does not explicitly* depend on the time t, i.e. mathematically that

$$\frac{partialL}{\partial t} = 0.$$

In this case we have

$$\begin{aligned}
\frac{dL}{dt} &= \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t} \\
&= \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right). \qquad (8.1)
\end{aligned}$$

Between the second and the third line we have used our hypothesis that the Lagrangian does not depend explicitly from the parameter t and in the last equality we have used that the equations of motion are satisfied. We thus get the equality

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right),$$

i.e.

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0.$$

Thus if the Lagrangian does not depend explicitly on time, the quantity

$$\dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

is an *integral of the motion*[1]

8.2 Conservation laws in a special relativistic field theory

Let us consider a theory consisting of N fields $\{\phi^{(i)}\}_{i=1,\dots,N}$, described by the Lagrangian density $\mathcal{L}(x^\mu, \phi^{(i)}, \partial_\nu \phi^{(j)})$. The dynamics of the theory is described by the Euler-Lagrange equations,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})}\right) = \frac{\partial \mathcal{L}}{\partial \phi^{(i)}} \quad i = 1, \dots, N.$$

Let us now make the additional hypothesis that the Lagrangian does not depend explicitly from x^μ, i.e.

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0.$$

In this case we have

$$\begin{aligned}
\partial_\nu \mathcal{L} &= \sum_i^{1,N} \frac{\partial \mathcal{L}}{\partial \phi^{(i)}} \partial_\nu \phi^{(i)} + \sum_i^{1,N} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})} \partial_\nu \partial_\mu \phi^{(i)} \\
&= \sum_i^{1,N} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})}\right) \partial_\nu \phi^{(i)} + \sum_i^{1,N} \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^{(i)})} \partial_\mu \partial_\nu \phi^{(i)} \\
&= \sum_i^{1,N} \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})}\right) (\partial_\nu \phi^{(i)}) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^{(i)})} \partial_\mu (\partial_\nu \phi^{(i)})\right] \\
&= \sum_i^{1,N} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})} (\partial_\nu \phi^{(i)})\right) \\
&= \partial_\mu \sum_i^{1,N} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})} (\partial_\nu \phi^{(i)})\right). \qquad (8.2)
\end{aligned}$$

Again we remember our hypothesis that the dependence of $\mathcal{L}$ from x^μ is only through the fields $\phi^{(i)}$ and their derivatives in the first line. We then use the

[1] Actually, if we remember that

$$p = \frac{\partial L}{\partial \dot{q}}$$

we see that the conserved quantity is just the Hamiltonian of the system,

$$H = p\dot{q} - L.$$

field equations in the second line. The final result is then

$$\delta^{\mu}_{\nu}\partial_{\mu}\mathcal{L} = \partial_{\mu}\sum_{i}^{1,N}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^{(i)})}(\partial_{\nu}\phi^{(i)})\right),$$

or, which is the same,

$$\partial_{\mu}(\delta^{\mu}_{\nu}\mathcal{L}) = \partial_{\mu}\sum_{i}^{1,N}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^{(i)})}(\partial_{\nu}\phi^{(i)})\right),$$

so that

$$\partial_{\mu}T^{\mu}_{\nu} = 0,$$

where we have defined

$$T^{\mu}_{\nu} = \sum_{i}^{1,N}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^{(i)})}(\partial_{\nu}\phi^{(i)})\right) - \delta^{\mu}_{\nu}\mathcal{L}.$$

Definition 8.1 (Stress Energy Tensor)
Let us consider a Field Theory consisting of N fields $\phi^{(i)}$ in n dimensions, that admits a Lagrangian formulation in terms of a Lagrangian density $\mathcal{L}$. The quantity

$$T^{\mu}_{\nu} = \sum_{i}^{1,N}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi^{(i)})}(\partial_{\nu}\phi^{(i)})\right) - \delta^{\mu}_{\nu}\mathcal{L}$$

is called the Stress-Energy tensor *of the fields.*

Proposition 8.1 (Local conservation laws)
If in the Lagrangian formulation of a field theory of N fields $\phi^{(i)}$ in n dimensions the Lagrangian density does not depend explicitly on the coordinates, the the stress-energy tensor is locally conserved,

$$\partial_{\mu}T^{\mu}_{\nu} = 0,$$

i.e. its divergence is zero.