Chapter 7

Special Relativity: Problems

7.1 Kinematics

Problem 7.1 (Relation between 4- and 3-velocity)
Let us consider the 4-velocity $\boldsymbol{u}$ corresponding to an ordinary 3-velocity $\vec{v}$. How can we write:

1. *u^0 as a function of $|\vec{v}|$?*
2. *u^j as a function of $\vec{v}$?*
3. *u^0 as a function of u^j?*
4. *$d/d\tau$ as a function of d/dt and $\vec{v}$?*
5. *v^j as a function of u^j?*
6. *$|\vec{v}|$ as a function of u^0?*

Solution:

We remember that the 4-velocity $\boldsymbol{u}$ is defined as

$$\boldsymbol{u} = \frac{d\boldsymbol{x}}{d\tau},$$

where $\boldsymbol{x} = (ct, \vec{x})$ and

$$-d\tau^2 = ds^2 = -c^2 dt^2 + d\vec{x}^2.$$

Thus

$$d\tau^2 = c^2 dt^2 \left(1 - \frac{1}{c^2}\frac{d\vec{x}^2}{dt^2}\right)$$

and

$$\begin{aligned} d\tau &= cdt \left(1 - \frac{1}{c^2}\frac{d\vec{x}^2}{dt^2}\right)^{1/2} \\ &= cdt \left(1 - \frac{\vec{v}^2}{c^2}\right)^{1/2} \end{aligned}$$

$$= cdt \left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{1/2}$$
$$= \frac{c}{\gamma} dt, \tag{7.1}$$

where

$$\gamma = \frac{1}{\left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{1/2}}. \tag{7.2}$$

Thus

$$\boldsymbol{u} = \frac{\gamma}{c} \frac{d}{dt}(ct, \vec{x}) = \frac{\gamma}{c}\left(c, \frac{d\vec{x}}{dt}\right) = \left(\gamma, \gamma\frac{\vec{v}}{c}\right), \tag{7.3}$$

and

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = \eta_{\mu\nu} u^\mu u^\nu = -\gamma^2 + \gamma^2 \frac{|\vec{v}|^2}{c^2} = -\gamma^2 \left(1 - \frac{|\vec{v}|^2}{c^2}\right) = -1 \tag{7.4}$$

With these preliminary definitions we can attack the other points.

1. This comes directly from (7.2) and (7.3), since
$$u^0 = \gamma = \frac{1}{\left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{1/2}}.$$

2. Again, from (7.3) we obtain,
$$u^j = \gamma \frac{v^j}{c} = \frac{v^j}{(c^2 - |\vec{v}|^2)^{1/2}} = \frac{v^j}{(c^2 - v^i v_i)^{1/2}}.$$

3. This relation comes from the fact that the modulus of the four velocity is -1 (equation (7.4)). Thus
$$-1 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle = -(u^0)^2 + u^j u_j$$
and
$$u^0 = (1 + u^j u_j)^{1/2}. \tag{7.5}$$

4. This result comes directly from (7.1) and, indeed, it has already been used in (7.3):
$$\frac{d}{d\tau} = \frac{\gamma}{c}\frac{d}{dt} = \frac{1}{c\left(1 - \frac{|\vec{v}|^2}{c^2}\right)^{1/2}} \frac{d}{dt}.$$

5. We start from the definition of the 4-velocity and observe that
$$v^j = \frac{c}{\gamma} u^j = \frac{c}{u^0} u^j;$$
then from the result found in 3., we obtain
$$v^j = \frac{c u^j}{\left(1 + u_k u^k\right)^{1/2}}.$$

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6. This last result can be obtained from the previous one and from (7.5)

$$\begin{aligned} |\vec{v}| &= (v^j v_j)^{1/2} \\ &= c\left(\frac{u^j u_j}{1+u^k u_k}\right)^{1/2} \\ &= \frac{c}{u^0}\left((u^0)^2-1\right)^{1/2}. \end{aligned}$$

It could be useful for the reader to rewrite these results in the units in which $c \equiv 1$.

□

Problem 7.2 (4-acceleration)
Let us consider the 4-acceleration $\boldsymbol{a}$ *of a given observer. Show that it has only three independent components. What is the relation of these components with the components of the ordinary acceleration* $\vec{a}$*, i.e. with the components of the acceleration measured by the observer in his local frame of reference with a "Newtonian accelerometer"? How can be written, in intrinsic form, the acceleration measured in the reference system of the observer?*

Solution:

The 4-acceleration is defined as

$$\boldsymbol{a} = \frac{d\boldsymbol{u}}{d\tau},$$

where $\boldsymbol{u}$ is the 4-velocity of the observer. Since

$$\langle \boldsymbol{u}, \boldsymbol{u} \rangle = -1$$

it follows that

$$0 = \frac{d}{d\tau}\langle \boldsymbol{u}, \boldsymbol{u}\rangle = \left\langle \frac{d\boldsymbol{u}}{d\tau}, \boldsymbol{u}\right\rangle + \left\langle \boldsymbol{u}, \frac{d\boldsymbol{u}}{d\tau}\right\rangle = 2\langle \boldsymbol{a}, \boldsymbol{u}\rangle,$$

so that the 4-acceleration is always orthogonal to the 4-velocity. Now think at an observer subjected to a 4-acceleration: its velocity is changing with time, i.e. it is not constant in any inertial frame. On the other hand, at a given instant of time, we can always find an inertial frame which has the same velocity as the observer: this is called a *locally comoving frame*, where the word "locally" emphasizes that it is comoving only at a given instant of time, i.e. at a given place along the trajectory of the observer. In this frame the observer has 4-velocity $\hat{\boldsymbol{u}} = (-1, \vec{0})$, and the orthogonality condition reads

$$0 = \langle \boldsymbol{a}, \boldsymbol{u}\rangle = \left\langle (a^0, \vec{a}), (-1, 0)\right\rangle = -a^0$$

so that it implies $a^0 = 0$ and a^i arbitrary.

A Newtonian accelerometer could be obtained by letting the observer release a particle in the comoving frame and seeing how much velocity $d\vec{v}$ the observer gains relative to it in a short time $d\tau$. Then we could compute $\vec{a} = d\vec{v}/d\tau$. Of course the particle is really stationary in the momentarily comoving inertial frame and we accelerate relative to it by

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an amount $du^j = a^j d\tau$. Differentiating the expression for u^j in terms of v^j and setting $\vec{v} = 0$, we obtain

$$a^j_{\mathrm{New}} = a^j .$$

Thus the 3 independent components of the 4-acceleration in a comoving frame are just the 3-Newtonian acceleration. Moreover, since $\langle \boldsymbol{a}, \boldsymbol{u} \rangle = 0$ and in the observer local rest frame

$$\boldsymbol{a} = (0, \hat{a}^j),$$

with $\hat{a}^j$ the j-th component of the locally measured acceleration, then the squared magnitude of his locally measured acceleration is

$$a^2 = \hat{a}^j \hat{a}_j = \left\langle (0, \hat{a}^j), (0, \hat{a}^j) \right\rangle = \langle \boldsymbol{a}, \boldsymbol{a} \rangle .$$

□

Problem 7.3 *Let us consider two Lorentz transformations:*

1. *a boost with velocity v_x in the x direction;*
2. *a boost with velocity v_y in the y direction.*

What is the Lorentz matrix associated to the composition of the two transformations in the order given above? And what the Lorentz matrix if we invert the order?

Solution:

We write the Lorentz boost with velocity v_x along the x-axis:

$$\begin{cases} ct' &= \gamma_x (ct - \beta_x x) \\ x' &= \gamma_x (x - \beta_x ct) \\ y' &= y \\ z' &= z \end{cases} ,$$

where $\beta_x = v_x/c$ and $\gamma_x = (1 - \beta_x^2)^{-1/2}$. If we set $x^0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$ and the same for primed quantities, then the above transformation can be written $x'^\mu = (\Lambda_x)^\mu{}_\nu x^\nu$, with[1]

$$(\Lambda_x)^\mu{}_\nu = \begin{pmatrix} \gamma_x & -\gamma_x \beta_x & 0 & 0 \\ -\gamma_x \beta_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

If we instead consider a Lorentz boost with velocity v_y along the y axis, then, with analogous definitions and conventions,

$$\begin{cases} ct' &= \gamma_y (ct - \beta_y y) \\ x' &= x \\ y' &= \gamma_y (y - \beta_y ct) \\ z' &= z \end{cases} ,$$

[1]The symbol used here for the transformation matrix, $(\Lambda_x)^\mu{}_\nu$, can be related with the one used in changing the basis vectors, as in problem 4.1: in particular, if we set $(\Lambda_x) \equiv \Lambda$, we can write $\Lambda^\mu{}_\nu = (\Lambda^{-1})_\nu{}^\mu$.

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which is thus associated to the matrix

$$(\Lambda_y)^\mu{}_\nu = \begin{pmatrix} \gamma_y & 0 & -\gamma_y\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_y\beta_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we first perform the boost in the x direction and then the one in the y direction, the associated matrix is

$$\begin{aligned} \Lambda_{xy} &= \Lambda_y\Lambda_x \\ &= \begin{pmatrix} \gamma_x & -\gamma_x\beta_x & 0 & 0 \\ -\gamma_x\beta_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_y & 0 & -\gamma_y\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_y\beta_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_x\gamma_y & -\gamma_x\beta_x & -\gamma_x\gamma_y\beta_y & 0 \\ -\gamma_x\gamma_y\beta_x & \gamma_x & \gamma_x\gamma_y\beta_x\beta_y & 0 \\ -\gamma_y\beta_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Performing the transformations in the reverse order,

$$\begin{aligned} \Lambda_{yx} &= \Lambda_x\Lambda_y \\ &= \begin{pmatrix} \gamma_y & 0 & -\gamma_y\beta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_y\beta_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_x & -\gamma_x\beta_x & 0 & 0 \\ -\gamma_x\beta_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_x\gamma_y & -\gamma_x\gamma_y\beta_x & -\gamma_y\beta_y & 0 \\ -\gamma_x\beta_x & \gamma_x & 0 & 0 \\ -\gamma_x\gamma_y\beta_y & \gamma_x\gamma_y\beta_x\beta_y & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{7.6}$$

and we see that $\Lambda_{xy} \neq \Lambda_{yx}$.

□

Problem 7.4 *Let us consider two events S_1 and S_2 separated by a spacelike interval. Show that there exists a Lorentz frame where the two events are simultaneous, but there exists no Lorentz frame where the two events happen in the same place.*
Let us then consider two events T_1 and T_2 separated by a timelike interval. Show that there exists a Lorentz frame where the two events happen in the same place, but there exists no Lorentz frame where the two events are simultaneous.

Solution:

Let us consider S_1 (x^0, x^1, x^2, x^3) and S_2 $(x^0+\Delta x^0, x^1+\Delta x^1, x^2+\Delta x^2, x^3+\Delta x^3)$, so that their spacelike separation can be written as:

$$\Delta s^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2\Delta t^2 + \Delta l^2 = k^2 > 0.$$

In another Lorentz frame, since the interval is invariant under Lorentz transformations, we have

$$0 < k^2 = -c^2(\Delta t')^2 + (\Delta l')^2; \tag{7.7}$$

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if in this new reference frame the two events have to be simultaneous, then we must have $\Delta t' = 0$, which is consistent with equation (7.7) and shows that in this frame the two events will be separated by a distance $\Delta l' = k$. On the other hand in no Lorentz frame the two events can happen in the same place. Indeed, this would imply $\Delta l' = 0$ and thus from (7.7)

$$-c^2(\Delta t')^2 > 0,$$

which is impossible for real $\Delta t'$.

If the two events, now called T_1 (x^0, x^1, x^2, x^3) and T_2 ($x^0 + \Delta x^0$, $x^1 + \Delta x^1$, $x^2 + \Delta x^2$, $x^3 + \Delta x^3$), are timelike separated this implies

$$-c^2 \Delta t^2 + \Delta l^2 = -k^2 < 0$$

and after a Lorentz transformation, again for the invariance of the interval,

$$0 > -k^2 = -c^2(\Delta t')^2 + (\Delta l')^2. \tag{7.8}$$

Now the simultaneity of the two events, $\Delta t' = 0$, is in contrast with equation (7.8) and $\Delta l' \in \mathbb{R}$. On the other hand the two events can now happen at the same place. In this case $\Delta l' = 0$ gives no troubles in (7.8) and $\Delta t' = k/c$ is the time that an observer in this frames measures between the two events.

□

Problem 7.5 *Let us consider an observer which is at rest in the origin of a reference frame. Show in a diagram the set of all events which are simultaneous with him at the instant $t = 0$. Then, let us consider another observer, which translates uniformly and with velocity v with respect to the first one. Add to the previously drawn diagram his worldline and the set of all events which are simultaneous with him at $t = 0$.*

Solution:

See figure 7.1.

□

Problem 7.6 *Write the metric tensor and the connection symbols for:*

1. *Minkowski spacetime in cartesian coordinates;*
2. *Minkowski spacetime in polar coordinates;*
3. *a spherical surface in spherical coordinates.*

Solution:

As a premise, let us remember that equation (3.21), which gives the connection coefficients in terms of the metric, is not the most efficient way to compute them. We will anyway use that result, as well as the one obtained in problem 7.7.

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Figure 7.1: World lines and surfaces of simultaneity of observer in relative translational motion with constant velocity.

1. The metric tensor for Minkowski spacetime in cartesian coordinates is

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta^{\mu\nu}$$

and is constant. Thus all derivatives of the metric are vanishing and so are all the connection coefficients: $\Gamma^{\lambda}_{\mu\nu} \equiv 0$.

2. In polar coordinates the Minkowski spacetime line element can be written as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^{\mu} \otimes dx^{\nu} \\ &= -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \\ &= -dt^2 + dr^2 + r^2 d\Omega^2, \end{aligned}$$

where it is customary to denote with $d\Omega^2$ the element of solid angle

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

The metric tensor can then be written as

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix},$$

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with inverse

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

and we are in a coordinate basis. All the connection coefficients with three different indices vanish, in view of the results of problem 7.7. Those results also imply that, since the metric is *static* i.e. no time dependence appears, all time derivatives of the metric coefficients vanish and thus $\Gamma^0_{\mu(\mu)}$, $\Gamma^\lambda_{(\lambda)0}$ also vanish. Moreover, g_{00} and g_{11} are constant, so that all their derivatives vanish and with them Γ^λ_{00}, Γ^λ_{11}, $\Gamma^0_{0\nu}$, $\Gamma^1_{1\nu}$. The metric coefficients are also independent from ϕ, i.e. all derivatives of the metric coefficients with respect to x_3 vanish, so that $\Gamma^3_{\mu(\mu)} = \Gamma^\lambda_{(\lambda)3} = 0$. Using the fact that in a coordinate basis the connection coefficients are symmetric, we have to compute the only non-vanishing coefficients Γ^1_{22}, Γ^1_{33}, Γ^2_{12}, Γ^2_{33}, Γ^3_{13}, Γ^3_{23}. Using again the results of problem 7.7, we obtain

$$\begin{aligned} \Gamma^1_{22} &= -r & \Gamma^1_{33} &= -r\sin^2\theta \\ \Gamma^2_{12} &= \frac{1}{r} & \Gamma^2_{33} &= -\sin\theta\cos\theta \\ \Gamma^3_{13} &= \frac{1}{r} & \Gamma^3_{23} &= \cot\theta. \end{aligned} \tag{7.9}$$

3. We consider a spherical surface and adopt spherical coordinates:

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2,$$

so that the metric is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{pmatrix}, \quad \mu,\nu = 2,3$$

and we choose the natural coordinate basis. The metric is diagonal and ϕ independent, the connection coefficient are symmetric: we again will use the results of problem 7.7 to compute the non-vanishing elements, Γ^2_{33} and Γ^3_{23}:

$$\Gamma^2_{33} = -\sin\theta\cos\theta \qquad \Gamma^3_{23} = \cot\theta \tag{7.10}$$

Note that the coefficients of 3. are the same as the *angular* part of 2..

□

Problem 7.7 *Let us consider a diagonal metric in a coordinate basis. Show that:*

$$\Gamma^\lambda_{\mu\nu} = 0 \text{ if } \lambda \neq \mu \neq \nu \neq \lambda;$$

$$\Gamma^\lambda_{\mu(\mu)} = -\frac{1}{2g_{(\lambda)(\lambda)}}\partial_\lambda g_{\mu(\mu)} \text{ with } \lambda \neq \mu;$$

$$\Gamma^\lambda_{(\lambda)\nu} = \partial_\nu(\log|g_{\lambda(\lambda)}|^{1/2}) \text{ with } \lambda \neq \nu;$$

$\Gamma^{\lambda}_{(\lambda)(\lambda)} = \partial_{(\lambda)}(\log|g_{(\lambda)\lambda}|^{1/2})$.

Solution:

As a preliminary observation we observe that if a matrix is diagonal and non-singular,

$$A_{\mu\nu} = \begin{pmatrix} \heartsuit & 0 & 0 & 0 \\ 0 & \diamondsuit & 0 & 0 \\ 0 & 0 & \clubsuit & 0 \\ 0 & 0 & 0 & \spadesuit \end{pmatrix},$$

then its inverse is easily found

$$A^{\mu\nu} = \begin{pmatrix} \heartsuit^{-1} & 0 & 0 & 0 \\ 0 & \diamondsuit^{-1} & 0 & 0 \\ 0 & 0 & \clubsuit^{-1} & 0 \\ 0 & 0 & 0 & \spadesuit^{-1} \end{pmatrix}.$$

For our diagonal metric this means $g^{\mu(\mu)} = (g_{(\mu)\mu})^{-1}$. Now we attack the computation of the connection coefficients using their definition in a coordinate basis

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(-\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu}).$$

Since the metric is diagonal the summed index α must equal λ if we do not want a null factor in front of the round bracket, i.e. at most one term in the sum survives:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda(\lambda)}(-\partial_{(\lambda)} g_{\mu\nu} + \partial_\mu g_{\nu(\lambda)} + \partial_\nu g_{(\lambda)\mu}).$$

We thus see that if $\lambda \neq \mu \neq \nu \neq \lambda$, then in the round brackets all the g's are non-diagonal elements of the metric, i.e. they are null. Thus $\Gamma^{\lambda}_{\mu\nu} = 0$ if $\lambda \neq \mu \neq \nu \neq \lambda$. Since we have established that non-vanishing results require some of the three indices to be equal, we only need to analyze all these possibilities in turn:

$\mu = \nu \neq \lambda$: in this case only the first term in round brackets survives, and we have

$$\begin{aligned} \Gamma^{\lambda}_{\mu(\mu)} &= -\frac{1}{2} g^{\lambda(\lambda)} \partial_{(\lambda)} g_{\mu(\mu)} \\ &= -\frac{1}{2 g_{\lambda(\lambda)}} \partial_{(\lambda)} g_{\mu(\mu)}. \end{aligned}$$

$\lambda = \mu \neq \nu$: now the last term in round brackets survives and we have

$$\begin{aligned} \Gamma^{(\lambda)}_{\lambda\nu} = \Gamma^{(\lambda)}_{\nu\lambda} &= \frac{1}{2} g^{\lambda(\lambda)} \partial_\nu g_{(\lambda)(\lambda)} \\ &= \frac{1}{2} \frac{1}{g_{\lambda(\lambda)}} \partial_\nu g_{(\lambda)(\lambda)} \\ &= \frac{1}{2} \partial_\nu \log|g_{\lambda(\lambda)}| \\ &= \partial_\nu \log(|g_{\lambda(\lambda)}|^{1/2}). \end{aligned}$$

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$\lambda = \mu = \nu$: all three terms in round brackets now contribute, but two of them are opposite; thus we obtain

$$\begin{aligned} \Gamma^{\lambda}_{(\lambda)(\lambda)} &= \frac{1}{2} g^{(\lambda)(\lambda)} \partial_{\lambda} g_{(\lambda)(\lambda)} \\ &= \frac{1}{2} \frac{1}{g_{(\lambda)(\lambda)}} \partial_{\lambda} g_{(\lambda)(\lambda)} \\ &= \frac{1}{2} \partial_{\lambda} \log |g_{(\lambda)(\lambda)}| \\ &= \partial_{\lambda} \log(|g_{(\lambda)(\lambda)}|^{1/2}). \end{aligned}$$

We remember that indices in round brackets are not summed over!

□

Problem 7.8 *Show that the character of a geodesic cannot change along its path.*

Solution:

These is nothing but a consequence of the compatibility condition of the covariant derivative with the metric. Indeed if σ is a geodesic and $\dot{\boldsymbol{\sigma}}$ its tangent vector, from the definition of geodesic we know that $D\dot{\boldsymbol{\sigma}}/dt = 0$, i.e. $\dot{\boldsymbol{\sigma}}$ is parallel along σ. Thus if we consider $\langle \dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\sigma}} \rangle$ we have

$$\frac{d}{dt} \langle \dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\sigma}} \rangle = \left\langle \frac{D\dot{\boldsymbol{\sigma}}}{dt}, \dot{\boldsymbol{\sigma}} \right\rangle + \left\langle \dot{\boldsymbol{\sigma}}, \frac{D\dot{\boldsymbol{\sigma}}}{dt} \right\rangle = 2 \left\langle \frac{D\dot{\boldsymbol{\sigma}}}{dt}, \dot{\boldsymbol{\sigma}} \right\rangle = \langle \mathbf{0}, \dot{\boldsymbol{\sigma}} \rangle = 0$$

so that if the tangent vector is timelike/spacelike/null at one point (i.e. if the geodesic is timelike/spacelike/null at one point) it remains timelike/spacelike/null at all other points.

□

Problem 7.9 *Let us consider a* 2*-sphere,*

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2,$$

and a vector $\boldsymbol{A}$ *which is* $\boldsymbol{A} = \boldsymbol{e}_{\theta}$ *at the point* $(\theta = \theta_0, \phi = 0)$*. How does* $\boldsymbol{A}$ *change after a parallel transport along* $\theta = \theta_0$*? How does its modulus change?*

Solution:

Let us consider the vector $\boldsymbol{A} = \boldsymbol{e}_{\theta}$. Since it is parallel propagated along a line $\theta =$ const., then it is parallel propagated in the $\boldsymbol{e}_{\phi}$ direction (i.e we can consider parallel propagation along the curve $\theta = \theta_0$, $\phi \in [0, 2\pi]$, with tangent vector $\boldsymbol{e}_{\phi}$). The local condition for parallel propagation can be written as

$$D(\boldsymbol{e}_{\phi}, \boldsymbol{A}) = 0 \Longleftrightarrow A^{j}{}_{;\phi} = \partial_{\phi} A^{j} + \Gamma^{j}_{k\phi} A^{k} = 0.$$

The only non-vanishing connection coefficients, calculated in problem 7.6, are $\Gamma^{\theta}_{\phi\phi}$ and $\Gamma^{\phi}_{\theta\phi}$, so the equation above (set $\{j, k\}$=$\{\theta, \phi\}$) gives

$$\begin{aligned} \partial_{\phi} A^{\theta} - \sin\theta \cos\theta A^{\phi} &= 0 \\ \partial_{\phi} A^{\phi} + \cot\theta A^{\theta} &= 0. \end{aligned}$$

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Remember that θ is constant (we are along the previously defined path): differentiating the first equation of the system above we get

$$\partial_\phi^2 A^\theta - \sin\theta\cos\theta\partial_\phi A^\phi$$

and substituting the second for $\partial_\phi A^\phi$ we obtain

$$\partial_\phi^2 A^\theta + \cos^2\theta A^\theta = 0,$$

which can be solved as

$$A^\theta = H\cos(\phi\cos\theta) + K\sin(\phi\cos\theta),$$

with H, K, constants. Substituting the ∂_ϕ derivative of this solution in the first differential equation of the system above, we also find

$$A^\phi = -H\sin(\phi\cos\theta)/\sin\theta + K\cos(\phi\cos\theta)/\sin\theta.$$

Since when $\phi = 0$ we have $\boldsymbol{A} = \boldsymbol{e}_\theta$, i.e. $A^\theta = 1$ and $A^\phi = 0$, this implies $H = 1$ and $K = 0$, so that

$$\boldsymbol{A}(\phi) = (\cos(\phi\cos\theta), -\sin(\phi\cos\theta)/\sin\theta).$$

After parallel transportation around the circle, so when $\phi = 2\pi$, we obtain

$$\boldsymbol{A}(2\pi) = (\cos(2\pi\cos\theta), -\sin(2\pi\cos\theta)/\sin\theta) \neq \boldsymbol{e}_\theta,$$

so that the vector has changed. On the other hand its modulus is

$$\langle \boldsymbol{A}(2\pi), \boldsymbol{A}(2\pi)\rangle = \cos^2(2\pi\cos\theta) + \sin^2(2\pi\cos\theta) = 1 = \langle \boldsymbol{A}(0), \boldsymbol{A}(0)\rangle\,;$$

thus it is unchanged.

□

Problem 7.10 *Show that in a coordinate basis* $\Gamma_{\alpha\beta\gamma}$ *is symmetric in the indices* β, γ.
Show that in an orthonormal basis $\Gamma_{\alpha\beta\gamma}$ *is antisymmetric in the indices* α, γ.
Show that in an arbitrary basis the connection has an antisymmetric part.

Solution:

For the first part let us fix a coordinate basis

$$\left\{\frac{\partial}{\partial x_i}\right\}_{i=1,\ldots,m}.$$

We remember that in a coordinate basis

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

so that the fact that a connection is symmetric implies

$$D(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) - D(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}) = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0.$$

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Thus

$$\begin{aligned} D(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) &= D(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}) \\ \Rightarrow \Gamma^k_{ij} \frac{\partial}{\partial x_k} &= \Gamma^k_{ji} \frac{\partial}{\partial x_k} \end{aligned}$$

and, since the basis element are linear independent,

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

Let us then consider an orthonormal basis $\{\boldsymbol{e}_\mu\}$ and remember that

$$\Gamma_{\alpha\beta\gamma} = \langle \boldsymbol{e}_\alpha, D(\boldsymbol{e}_\beta, \boldsymbol{e}_\gamma) \rangle = g_{\alpha\mu} \Gamma^\mu_{\beta\gamma},$$

where in this basis $g_{\rho\sigma} = \delta_{\rho\sigma}$. We thus have

$$\begin{aligned} \Gamma_{\alpha\beta\gamma} + \Gamma_{\gamma\beta\alpha} &= \langle \boldsymbol{e}_\alpha, D(\boldsymbol{e}_\beta, \boldsymbol{e}_\gamma) \rangle + \langle \boldsymbol{e}_\gamma, D(\boldsymbol{e}_\beta, \boldsymbol{e}_\alpha) \rangle \\ &= \boldsymbol{e}_\beta(\langle \boldsymbol{e}_\alpha, \boldsymbol{e}_\gamma \rangle) \\ &= \boldsymbol{e}_\beta(\delta_{\alpha\gamma}) \\ &= 0. \end{aligned}$$

In a generic basis the presence of an antisymmetric part in the connection coefficients can be traced back to the relation

$$D(\boldsymbol{V}, \boldsymbol{W}) - D(\boldsymbol{W}, \boldsymbol{V}) = [\boldsymbol{V}, W]$$

since $[-, -]$ is antisymmetric.

□

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