Chapter 6

Special Relativity

6.1 The group of Lorentz transformations

6.1.1 2-dimensional case

Let us consider the invariant interval defined in our derivation of Lorentz transformations in the previous chapter. In particular let us consider preliminarily the 2-dimensional case, in which the finite invariant interval can be written as

$$s^2 = x^2 - t^2.$$

If in $\mathbb{R}^2$ we take the vector $\boldsymbol{x} = (t, x)$ and we equip the vector space of all these vectors with the pseudo-Euclidean structure defined by the scalar product

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = g_{AB} x^A x^B \quad , \quad A = 1, 2 \quad , \quad B = 1, 2,$$

where $g_{00} = -1$, $g_{01} = g_{10} = 0$ and $g_{11} = +1$. Requiring the invariance of the interval is tantamount of requiring the invariance of the pseudo-Euclidean structure, i.e. we are interesting of determining the general form of a linear transformation Λ such that

$$\boldsymbol{g} = \Lambda^T \boldsymbol{g} \Lambda.$$

From the validity of the above equation we know that the 2×2 matrix Λ is subject to the constraint

$$\det(\boldsymbol{g}) = \det(\Lambda^T \boldsymbol{g} \Lambda) = \det(\Lambda^T) \det(\boldsymbol{g}) \det(\Lambda)$$

which, since $\det(\Lambda) = \det(\Lambda^T)$, gives

$$\det(\Lambda)^2 = 1 \quad \Rightarrow \quad \det(\Lambda) = \epsilon \overset{\text{def.}}{=} \pm 1.$$

Moreover from the invariance of $\boldsymbol{g}$, if we set

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we obtain:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and performing the matrix multiplications on the right hand side

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c^2 - a^2 & -ab + cd \\ -ab + cd & b^2 - d^2 \end{pmatrix},$$

which, together with the constraint on the determinant

$$\det(\Lambda) = ad - bc = \epsilon,$$

we can rewrite as a system of four equations in four unknowns:

$$\begin{cases} a^2 - c^2 = 1 \\ cd - ab = 0 \\ b^2 - d^2 = 1 \\ ad - bc = \epsilon \end{cases} . \tag{6.1}$$

Note that, of course, the last equation is dependent from the other three. Thus only three parameters can be determined independently, or more precisely, the solution is going to be a one parameter family of transformations. In what follows we will call with capital letters the signs of the four parameters a, b, c, d, so that

$$\begin{aligned} a = A|a| \quad &, \quad b = B|b| \\ c = C|c| \quad &, \quad d = D|d| \end{aligned}$$

Let us set some constraints on them, as a preliminary step:

1. from the first equation we see that $a \neq 0$.
2. from the third equation we see that $d \neq 0$.
3. for the signs the equations, respectively, imply:

$$\begin{cases} A \neq 0 \\ AB = CD \\ D \neq 0 \\ AD - BC = \epsilon \end{cases} . \tag{6.2}$$

Let us now solve the first equation for a, the third for d and substitute in the second[1]:

$$\begin{cases} a = A\sqrt{1 + c^2} \\ AB|b|\sqrt{1 + c^2} = CD|c|\sqrt{1 + b^2} \\ d = D\sqrt{1 + b^2} \\ ad - bc = \epsilon \end{cases}$$

Using the second equation in (6.2) the second equation above can be simplified an squared to obtain $|b| = |c|$ as a solution. This can be rewritten as $b = \eta c$, where $\eta \stackrel{\text{def.}}{=} -1, 0, +1$. Using this relation in the third equation we also find $|a| = |d|$, so that we end up with the system:

$$\begin{cases} a = A\sqrt{1 + c^2} \\ b = \eta c \\ |d| = |a| \\ ad - bc = \epsilon \end{cases}$$

[1] Square roots are always *arithmetic* i.e. their sign is always positive.

Let us now rewrite the last equation in the above system in a different way that we are going to use later on. First we have

$$\begin{aligned} ad - bc &= AD|a||d| - BC|b||c| \\ &= ADa^2 - BCc^2 \\ &= AD(1+c^2) - BCc^2 \\ &= (AD - BC)c^2 + AD \end{aligned} \tag{6.3}$$

Case $\eta = 0$.

In this case $B = C = 0$, i.e. $b = c = 0$. Then $a = A$ and $d = D$ and there can be a sign difference between a and d. This is consistent the fourth equation, which exactly gives $AD = \epsilon$. Thus we obtain

$$\Lambda = A \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}.$$

Making the signs appear explicitly we obtain 4 matrix, the identity and four discrete transformations, as follows:

$$\begin{aligned} \text{Identity} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \text{Time reflection} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ \text{Space reflection} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \text{Space time reflection} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Case $\eta \neq 0$.

In this case $B = \pm 1$ and $C = \pm 1$. We can multiply the second equation in the system (6.2), relating the signs, by A and C, since now both are different from zero, to get $AD = BC$, i.e. $AD - BC = 0$. Substituting this identity in (6.3) we obtain again

$$AD = \epsilon.$$

Since $AD = BC$ and $BC = \eta$ we see that $\epsilon = \eta$ and, so that the fourth equation (6.1) is again a consequence of the three others. We are going to use ϵ in place of η in what follows, i.e. $b = \epsilon c$. The remaining three equations in (6.1) do not allow an unique solution of the system. Let us parametrize the family of solutions using $\beta = c/a$ (remember $a \neq 0$ always). Then we can rewrite the first three equations of (6.1) as

$$\begin{cases} 1 - \beta^2 = a^{-2} \\ \beta = b/d \\ b^2 - d^2 = 1 \end{cases}$$

This gives

$$\begin{cases} |a| = \left(1 - \beta^2\right)^{-\frac{1}{2}} \\ |b| = |\beta||d| = |c| \\ |d| = |a| \end{cases},$$

so that

$$\Lambda = \begin{pmatrix} A\left(1-\beta^2\right)^{-\frac{1}{2}} & B|\beta|\left(1-\beta^2\right)^{-\frac{1}{2}} \\ C|\beta|\left(1-\beta^2\right)^{-\frac{1}{2}} & D\left(1-\beta^2\right)^{-\frac{1}{2}} \end{pmatrix}.$$

From the above relation we factor the sign of a

$$\Lambda = A\begin{pmatrix} \left(1-\beta^2\right)^{-\frac{1}{2}} & AB|\beta|\left(1-\beta^2\right)^{-\frac{1}{2}} \\ AC|\beta|\left(1-\beta^2\right)^{-\frac{1}{2}} & AD\left(1-\beta^2\right)^{-\frac{1}{2}} \end{pmatrix}.$$

We can then fix the signs using previous results with the addition that sign $(\beta) = AC$:

$$\begin{cases} A = \epsilon D \\ B = \epsilon C \\ AB = CD \quad \Rightarrow \quad AD = BC \\ \text{sign}\,(\beta) = AC \end{cases}.$$

This gives

$$\Lambda = A\begin{pmatrix} \left(1-\beta^2\right)^{-\frac{1}{2}} & \epsilon\,\text{sign}\,(\beta)\,|\beta|\left(1-\beta^2\right)^{-\frac{1}{2}} \\ \text{sign}\,(\beta)\,|\beta|\left(1-\beta^2\right)^{-\frac{1}{2}} & \epsilon\left(1-\beta^2\right)^{-\frac{1}{2}} \end{pmatrix}$$

and we can conclude

$$\Lambda = A\begin{pmatrix} \left(1-\beta^2\right)^{-\frac{1}{2}} & \epsilon\beta\left(1-\beta^2\right)^{-\frac{1}{2}} \\ \beta\left(1-\beta^2\right)^{-\frac{1}{2}} & \epsilon\left(1-\beta^2\right)^{-\frac{1}{2}} \end{pmatrix}.$$

Although this result has been obtained when $\beta \neq 0$, it reproduces for $\beta = 0$ the identity matrix or the reflections obtained above. We will adhere to the convention

$$\gamma = \left(1-\beta^2\right)^{-\frac{1}{2}}.$$

The set

$$\left\{\Lambda \middle| \Lambda = A\begin{pmatrix} \gamma & \epsilon\gamma\beta \\ \gamma\beta & \epsilon\gamma \end{pmatrix}, A = \pm 1, \epsilon = \pm 1, -1 \leq \beta \leq 1\right\}$$

equipped with matrix multiplication is a group, the *Lorentz group*.

6.2 Accelerated Observers in Minkowski spacetime

Let us consider a 2-dimensional Minkowski spacetime

$$ds^2 = \boldsymbol{g} = \boldsymbol{\eta} = \eta_{\mu\nu} dx^\mu \otimes dx^\nu = -dt^2 + dx^2.$$

Let us consider an observer stationary at the origin $x = 0$ and let $L_{(0)}$ be his world-line. At $t = 0$ all the events which are simultaneous with him are those which satisfy the equation $t = 0$, i.e. they are the points on the x-axis. We will now apply to these events, $\mathtt{E}_0^{(\rho)} = (0, \rho)$, the boosts about O, which can be written as

$$\begin{cases} t' &= \gamma(t + \beta x/c) \\ x &= \gamma(x + \beta t) \end{cases},$$

where as usual

$$\beta = \frac{v}{c} \quad \text{and} \quad \gamma = \left(1 - \beta^2\right)^{-1/2}.$$

If we restrict our attention to one of the $\mathtt{E}_0^{(\rho)}$, the locus of the points that can be obtained by all possible boosts is given by the points of the hyperbola

$$x^2 - t^2 = \rho^2$$

with $x > 0$. The reason for this is that the interval

$$\Delta s^2 = \Delta x^2 - \Delta t^2$$

is invariant under a Lorentz transformation and it equals ρ^2 for the segment $\overline{\mathtt{OE}_0^{\rho}}$. Thus all the points on the curve

$$L^{(\rho)} = \left\{(t, x)|x^2 - t^2 = \rho^2, x > 0\right\}$$

can be parametrized by the quantity β which appears in the Lorentz transformation and are of the form $(\gamma\beta\rho/c, \gamma\rho)$. Note that only for $\rho > 0$ the Lorentz transformation defines a world-line starting from $\mathtt{E}_0^{(\rho)}$, since if $rho = 0$, $(0, 0)$ is a fixed point of them. We want now study some properties of Minkowski spacetime, with respect to observers with world-lines $L^{(\rho)}$.

Observers on $L^{(\rho)}$ are at constant distance from each other.

To prove this fact let us choose two events, $\mathtt{E}_0^{(\rho_1)}$ and $\mathtt{E}_0^{(\rho_2)}$ at $t = 0$. As seen from O they are separated by a distance $\Delta l = |\rho_2 - \rho_1|$. For an observer on $L^{(\rho)}$ which has speed proportional to the parameter β at $\mathtt{E}_0^{(\rho)}$ which is $\beta = 0$, the distance between the two events is the same. Now we consider the points which are obtained for a β parameter distance $\Delta\beta$, i.e. $\mathtt{E}_{\Delta\beta}^{(\rho_1)}$ and $\mathtt{E}_{\Delta\beta}^{(\rho_2)}$. For these points we have

$$\mathtt{E}_{\Delta\beta}^{(\rho_1)} = (\gamma_{\Delta\beta}(\Delta\beta)\rho_1/c, \gamma_{\Delta\beta}\rho_1) \quad \text{and} \quad \mathtt{E}_{\Delta\beta}^{(\rho_1)} = (\gamma_{\Delta\beta}(\Delta\beta)\rho_2/c, \gamma_{\Delta\beta}\rho_2).$$

There distance for the observer $L^{(0)}$ is now

$$\Delta l = \gamma_{\Delta\beta}|\rho^2 - \rho^1|$$

but for the observer on $L^{(\rho)}$, which is characterized by a velocity proportional to $\Delta\beta$, the distance Δl is contracted by a factor $1/\gamma_{\Delta\beta}$, i.e. it is $|\rho_2 - \rho_1|$ again.

Observers on $L^{(\rho)}$ are uniformly accelerated.

Let us choose two events $\mathtt{E}_{\beta}'^{(\rho)}$ and $\mathtt{E}_{\beta+\Delta\beta}^{(\rho)}$, on the world-line of the observer $L^{(\rho)}$. Let the two events be characterized by the following coordinate sets,

$$\begin{aligned} E_{\beta+\Delta\beta}^{(\rho)} &= (t, x) \\ E_{\beta}'^{(\rho)} &= (t', x'), \end{aligned}$$

where by definition of the world-line $L^{(\rho)}$, i.e. of the fundamental observer on it,

$$t = \gamma_{\Delta\beta}(t' + x'\Delta\beta) \quad \text{and} \quad x = \gamma_{\Delta\beta}(x' + t'\Delta\beta).$$

The proper time $\Delta\tau$ between the two events satisfies

$$-\Delta\tau^2 = -\Delta t^2 + \Delta x^2$$

where

$$\Delta t = t - t' \quad \text{and} \quad \Delta x = x - x'.$$

Thus

$$\begin{aligned}
\Delta\tau^2 &= \left(t' - \gamma_{\Delta\beta}(t' + x'\Delta\beta)\right)^2 - \left(x' - \gamma_{\Delta\beta}(x' + t'\Delta\beta)\right)^2 \\
&= (t')^2 - 2(t')\gamma_{\Delta\beta}(t' + x'\Delta\beta) + (\gamma_{\Delta\beta})^2(t' + x'\Delta\beta)^2 + \\
&\qquad -(x')^2 + 2(x')\gamma_{\Delta\beta}(x' + t'\Delta\beta) - (\gamma_{\Delta\beta})^2(x' + t'\Delta\beta)^2 \\
&= (t')^2 - (x')^2 - 2(t')^2\gamma_{\Delta\beta} - 2t'x'\Delta\beta\gamma_{\Delta\beta} + 2(x')^2\gamma_{\Delta\beta} + 2t'x'\Delta\beta\gamma_{\Delta\beta} \\
&\qquad +(\gamma_{\Delta\beta})^2\left[(t')^2 + 2x't'\Delta\beta + (x')^2(\Delta\beta)^2 - (x')^2 - 2x't'\Delta\beta - (t')^2(\Delta\beta)^2\right] \\
&= \left[(t')^2 - (x')^2\right] - 2\gamma_{\Delta\beta}\left[(t')^2 - (x')^2\right] + (\gamma_{\Delta\beta})^2\left[1 - (\Delta\beta)^2\right]\left[(t')^2 - (x')^2\right] \\
&= \left[(t')^2 - (x')^2\right]\left(1 - 2\gamma_{\Delta\beta} + (\gamma_{\Delta\beta})^2 - (\gamma_{\Delta\beta})^2(\Delta\beta)^2\right) \\
&= \left[(t')^2 - (x')^2\right]\left(1 - 2\gamma_{\Delta\beta} + (\gamma_{\Delta\beta})^2(1 - (\Delta\beta)^2)\right) \\
&= 2\left[(x')^2 - (t')^2\right](\gamma_{\Delta\beta} - 1) \\
&= 2\rho^2(\gamma_{\Delta\beta} - 1),
\end{aligned}$$

where, since $\mathtt{E}_\beta^{(\rho)}$ is on $L^{(\rho)}$, we have used that $(x')^2 - (t')^2 = \rho^2$. When $\Delta\beta \ll 1$ we have $\gamma_{\Delta\beta} \approx 1 - \Delta v^2/2$ and the above relation can be written as

$$\Delta\tau^2 \approx 2\rho^2\frac{1}{2}\Delta\beta^2$$

or in infinitesimal form

$$d\tau^2 = \rho^2\delta\beta^2.$$

In this expression $\Delta\beta$ is the increase in velocity that takes place between two infinitesimally close events, between which the observer on $L^{(\rho)}$ measures a time lapse $d\tau$. Thus an observer on $L^{(\rho)}$ measures an instantaneous acceleration

$$a = \frac{d\beta}{d\tau} = \frac{1}{\rho}$$

i.e. it is uniformly accelerated.

Red-shift by fundamental observers.

We will now compute the red-shift due to the relative acceleration of two observers moving on world lines $L^{(\rho_1)}$ and $L^{(\rho_2)}$ respectively. We remember that the red-shift is defined as

$$z = \frac{\lambda_{\text{Received}} - \lambda_{\text{Emitted}}}{\lambda_{\text{Emitted}}}$$

with

$$\lambda = c\Delta\tau.$$

Thus we have

$$z = \frac{\lambda_{\text{Received}}}{\lambda_{\text{Emitted}}} - 1 = \frac{\Delta\tau_{\text{Receiver}}}{\Delta\tau_{\text{Emitter}}} - 1,$$

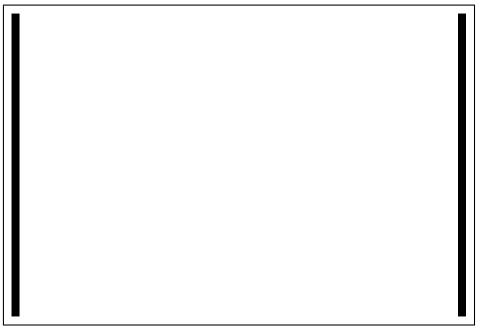

Figure 6.1: Red-shift between fundamental observers.

and we see that what really matters is how a time interval on the emitter world-line is measured from the receiver one. In our case a signal emitted in a parameter lapse $\Delta\beta$ from $L^{(\rho_1)}$ is such that

$$\Delta\tau_1^2 = 2\rho_1^2(\gamma_{\Delta\beta} - 1)$$

whereas on the receiver world-line $L^{(\rho_2)}$ we have

$$\Delta\tau_2^2 = 2\rho_2^2(\gamma_{\Delta\beta} - 1).$$

Thus

$$1 + z = \frac{\Delta\tau_{\text{Receiver}}}{\Delta\tau_{\text{Emitter}}} = \frac{\Delta\tau_2}{\Delta\tau_1} = \frac{\rho_2}{\rho_1}.$$

Red-shift by a stationary observer.

We are now interested in the shift experienced by the stationary observer on $L^{(0)}$ when he receives a signal from an observer $L^{(\rho)}$. Of course a parameter lapse $\Delta\beta$ again corresponds on $L^{(\rho)}$ to a proper time interval

$$\Delta\tau^2 = 2\rho^2(\gamma_{\Delta\beta} - 1).$$

We need now to know how is the $\Delta\tau'$ measured on $L^{(0)}$. With reference to figure 6.2 we see this interval can be computed as (in units where $c \equiv 1$)

$$\Delta\tau' = \overline{AB} = (t_{B''} - \overline{B'B''}) - (t_{A''} - \overline{A'A''}).$$

We set

$$A' = (t, x_{A'}) = (\gamma\beta\rho, \gamma\rho)$$

where as usual $\gamma = (1 - \beta^2)^{-1/2}$ and t is the time at which the signal arrives at A'. Moreover B' is a parameter distance $\Delta\beta$ along $L^{(\rho)}$, which means it can be obtained with a Lorentz transformation from A' with velocity $\Delta\beta$:

$$B' = (\gamma_{\Delta\beta}(\gamma\beta\rho + \Delta\beta\gamma\rho), \gamma_{\Delta\beta}(\gamma\rho + \Delta\beta\gamma\beta\rho)).$$

From the definition of A', since we have $\gamma\beta\rho = t$, using the definition of

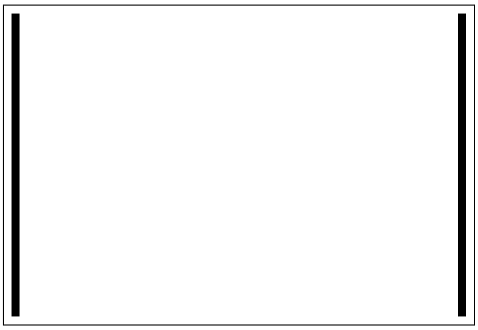

Figure 6.2: Red-shift by a stationary observer.

γ we can derive the following equalities,

$$\begin{aligned} v &= \frac{t}{(t^2+\rho^2)^{1/2}}, \\ \gamma &= \frac{(t^2+\rho^2)^{1/2}}{\rho}, \end{aligned} \tag{6.4}$$

which are useful to express the coordinates of A' and B' solely in terms of t, ρ and $\Delta\beta$:

$$\begin{aligned} A' &= (t, (t^2+\rho^2)^{1/2}) \\ B' &= (\gamma_{\Delta\beta}(t+\Delta\beta(t^2+\rho^2)^{1/2})), \gamma_{\Delta\beta}((t^2+\rho^2)^{1/2}) + \Delta\beta t). \end{aligned}$$

Using these results we now get

$$\begin{aligned} \overline{AB} &= \gamma_{\Delta\beta}(t+\Delta\beta(t^2+\rho^2)^{1/2})) - t + \\ &\qquad -\gamma_{\Delta\beta}((t^2+\rho^2)^{1/2}) + \Delta\beta t) + (t^2+\rho^2)^{1/2} \\ &= \gamma_{\Delta\beta} t + \gamma_{\Delta\beta}\Delta\beta(t^2+\rho^2)^{1/2} - t + \\ &\qquad -\gamma_{\Delta\beta}(t^2+\rho^2)^{1/2} - \gamma_{\Delta\beta}\Delta\beta t + (t^2+\rho^2)^{1/2} \\ &= t(\gamma_{\Delta\beta} - 1 - \gamma_{\Delta\beta}\Delta\beta) - (t^2+\rho^2)^{1/2}(\gamma_{\Delta\beta} - 1 - \gamma_{\Delta\beta}\Delta\beta) \\ &= (t - (t^2+\rho^2)^{1/2})(\gamma_{\Delta\beta} - 1 - \gamma_{\Delta\beta}\Delta\beta) \\ &= (\gamma_{\Delta\beta}(\Delta\beta - 1))((t^2+\rho^2)^{1/2} - t) \\ &= \frac{(1-(1-\Delta\beta)\gamma_{\Delta\beta})\rho^2}{t+(t^2+\rho^2)^{1/2}}; \end{aligned} \tag{6.5}$$

thus

$$\Delta\tau' = \frac{(1-(1-\Delta\beta)\gamma_{\Delta\beta})\rho^2}{t+(t^2+\rho^2)^{1/2}}.$$

When $\Delta\beta \ll 1$ we have the natural approximations

$$\gamma_{\Delta\beta} = (1-\Delta\beta^2)^{-1/2} \approx 1 + \frac{\Delta\beta^2}{2}$$

and

$$(\gamma_{\Delta\beta} - 1)^{1/2} \approx \frac{\Delta\beta}{\sqrt{2}}.$$

Using them we get

$$
\begin{aligned}
1+z &= \frac{\Delta\tau}{\Delta\tau'} \\
&= \frac{\sqrt{2}\rho(\gamma_{\Delta\beta}-1)^{1/2}\left[t+(\rho^2+t^2)^{1/2}\right]}{\rho^2\left[1-(1-\Delta\beta)\gamma_{\Delta\beta}\right]} \\
&\approx \frac{t+(\rho^2+t^2)^{1/2}}{\rho} \\
&\approx \frac{t}{\rho}+\left[1+\left(\frac{t}{\rho}\right)^2\right]^{1/2} .
\end{aligned}
$$