Chapter 1

Preliminaries

1.1 Linear Algebra preliminaries

1.2 Structures over a vector space

In this section V is a vector space of dimension $\dim(V) = n$. $\{e_1, \ldots, e_n\}$ is a basis of V and $\{E_1, \ldots, E_n\}$ a basis of V^*.

1.2.1 Exterior algebra

Let V be a vector space of dimension $\dim(V) = n$.

Definition 1.1 (k-linear alternating maps)
The space of k-linear alternating maps over V is the set

$$\Lambda^k(V) = \{\boldsymbol{\omega} | \boldsymbol{\omega} : V^k \longrightarrow \mathbb{R} \quad \text{with} \\ \boldsymbol{\omega}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = (-)^{\pi} \boldsymbol{\omega}(\boldsymbol{v}_{\pi(1)}, \ldots, \boldsymbol{v}_{\pi(k)}) \quad \text{if} \quad \omega \in \mathscr{S}_n\}$$

Proposition 1.1 (Vector space structure of $\Lambda^k(V)$)
$\Lambda^k(V)$ has a vector space structure. Let $\mathcal{B} = (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ be a basis of V and $\mathsf{c} = (\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_k})$, with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ a subsystem extracted from the basis $\mathcal{B}$. There is exactly one k-linear alternating map

$$\boldsymbol{\omega}_c : V^k \longrightarrow \mathbb{R}$$

such that

1. $\boldsymbol{\omega}_{\mathsf{c}}(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_k}) = 1$;
2. $\boldsymbol{\omega}_{\mathsf{c}}(\boldsymbol{e}_{j_1}, \ldots, \boldsymbol{e}_{j_k}) = 0$ *if* $\{j_1, \ldots, j_k\} \neq \{j_1, \ldots, j_k\}$;

Proposition 1.2 (Basis of $\Lambda^k(V)$)
Let

$$\mathcal{B}_{\Lambda^k} = \{\boldsymbol{\omega}_{\mathsf{c}} | \mathsf{c} = (\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_k})\}$$

$\mathcal{B}_{\Lambda^k}$ is a basis of $\Lambda^k(V)$. The dimension of $\Lambda^k(V)$ is given by the binomial coefficient $\begin{pmatrix} n \\ k \end{pmatrix}$.

We set $\Lambda^0 \stackrel{\text{def.}}{=} \mathbb{R}$. Then $\Lambda^1 = V^*$ and $\Lambda^n = \mathbb{R}$. Moreover $\Lambda^j = 0$ for $j > n$.

Definition 1.2 (Exterior product in $\Lambda^k(V)$)
Let $\kappa \in \Lambda^k(V)$ and $\lambda \in \Lambda^l(V)$.

$$\wedge : \Lambda^k(V) \times \Lambda^l(V) \longrightarrow \Lambda^{k+l}(V)$$

such that

$$(\boldsymbol{\kappa} \wedge \boldsymbol{\lambda})(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \dots, \boldsymbol{v}_{k+l}) \stackrel{\text{def.}}{=}$$
$$= \frac{1}{(k+l)!} \sum_{\pi \in \mathscr{S}_{k+l}} (-1)^\pi \boldsymbol{\kappa}(\boldsymbol{v}_{\pi(1)}, \dots, \boldsymbol{v}_{\pi(k)}) \boldsymbol{\lambda}(\boldsymbol{v}_{\pi(k+1)}, \dots, \boldsymbol{v}_{\pi(k+l)})$$

is called the exterior product*.*

The exterior product has the following properties:

1. if $\boldsymbol{\kappa} \in \Lambda^k(V)$ and $\boldsymbol{\lambda} \in \Lambda^l(V)$ then $\boldsymbol{\kappa} \wedge \boldsymbol{\tau} = (-1)^{kl} \boldsymbol{\tau} \wedge \boldsymbol{\kappa}$;
2. if $\boldsymbol{\omega} \in \Lambda^{2k+1}(V)$ then $\boldsymbol{\omega} \wedge \boldsymbol{\omega} = 0$.

Definition 1.3 (Graßmann Algebra of V)
The set

$$\mathcal{G}(V) = \bigoplus_{k}^{0,n} \Lambda^k(V)$$

together with the operations $(+, \cdot, \wedge)$ (vector space sum, vector space product by a scalar and exterior product) is an algebra with unity $1 \in \mathbb{R} \equiv \Lambda^0(V)$ $(1 \wedge \omega = \omega \wedge 1 = \omega)$, the Graßmann Algebra *over V.*

A basis of $\Lambda^k(V)$ can be written as

$$\mathcal{B}_{\Lambda^k} = \{\boldsymbol{E}_{i_1} \wedge \dots \wedge \boldsymbol{E}_{i_k} | 1 \le i_1 < i_2 < \dots < i_k \le n\}.$$

We can extend the exterior product as an operation on the Graßman algebra over a vector space V.

1.2.2 Tensor algebra

In this subsection let V, W, U be finite dimensional vector spaces over a field $\mathbb{F}$ (for definiteness $\mathbb{F}$ can be taught as $\mathbb{R}$ or $\mathbb{C}$). Let $F(V, W)$ be the *free vector space* generated by all couples (v, w) with $v \in V$ and $w \in W$: thus $F(V, W)$ is the set of all finite linear combinations of couples (v, w). $R(V, W)$ will be the subspace of $F(V, W)$ spanned by the following elements:

$$\begin{aligned}
(v_1 + v_2, w) - (v_1, w) - (v_2, w) &\qquad v_1, v_2 \in V, \quad w \in W \\
(v, w_1 + w_2) - (v, w_1) - (v, w_2) &\qquad v \in V, \quad w_1, w_2 \in W \\
(\alpha v, w) - \alpha(v, w) &\qquad v \in V, \quad w \in W, \quad \alpha \in \mathbb{F} \\
(v, \alpha w) - \alpha(v, w) &\qquad v \in V, \quad w \in W, \quad \alpha \in \mathbb{F}
\end{aligned}$$

Definition 1.4 (Tensor product)
The tensor product *of two vector spaces V and W is the vector space $V \otimes W$ defined as*

$$V \otimes W \stackrel{\text{def.}}{=} F(V,W) \setminus R(V,W) \quad .$$

The equivalence class in $V \otimes W$ containing the element (v,w) is denoted as $v \otimes w$. We will call ϕ the canonical bilinear map

$$\phi : V \times W \longrightarrow V \otimes W$$

such that $\phi(v,w) = v \otimes w$.

Definition 1.5 (Universal factorization property)
Let ψ be a bilinear map

$$\psi : V \times W \longrightarrow U$$

We will say that the couple (U,ψ) has the universal factorization property *for $V \times W$ if $\forall S$, S vector space, and*

$$\forall f, f : V \times W \longrightarrow S$$

f bilinear, there exists a unique $\tilde{f}$

$$\tilde{f} : U \longrightarrow S$$

such that $f = \tilde{f} \circ \psi$.

Proposition 1.3 (Universal factorization property of the tensor product)

The couple $(V \otimes W, \phi)$ has the universal factorization property for $V \times W$. Moreover the couple $(V \otimes W, \phi)$ is unique in the sense that if another couple (Z, ζ) has the universal factorization property for $V \times W$, then there exists an isomorphism α

$$\alpha : V \otimes W \longrightarrow Z$$

such that $\zeta = \alpha \circ \phi$.

Proof:

Let S be any vector space and f a bilinear map

$$f : V \times W \longrightarrow S$$

Since $V \times W$ is a basis for $F(V,W)$, f can be extended by linearity to a unique map

$$f' : F(V,W) \longrightarrow S$$

by the rule

$$f'(\sum_i^{1,N} \lambda_i (v_i, w_i)) = \sum_i^{1,N} \lambda_i f(v_i, w_i).$$

Since f is bilinear $ker(f') \supset R(V,W)$[1]. This means that f' induces a well defined map f''

$$f'' : V \otimes W \longrightarrow S$$

such that[2] $f''(v \otimes w) = f'((v,w))$. By construction $f'' \circ \phi = f$ and f'' is unique since $\phi(V \times W)$ spans $V \otimes W$. This shows that the couple $(V \otimes W, \phi)$ has the universal factorization property for $V \times W$.
Let us consider another couple (Z, ζ) having the universal factorization property for $V \times W$. When in the definition of the universal factorization property we use the following identifications

$$\begin{array}{ll} \psi \longleftrightarrow \phi & U \longleftrightarrow V \otimes W \\ f \longleftrightarrow \zeta & S \longleftrightarrow Z \end{array}$$

we obtain the existence of a unique map σ_1,

$$\sigma_1 : V \otimes W \longrightarrow Z$$

such that $\zeta = \sigma_1 \circ \phi$.
At the same time we can exchange the roles of $(U \otimes V, \phi)$ and (Z, ζ). This means that in the definition of the universal factorization property we can also use the following identifications

$$\begin{array}{ll} \psi \longleftrightarrow \zeta & U \longleftrightarrow Z \\ f \longleftrightarrow \phi & S \longleftrightarrow V \otimes W \end{array}$$

so that it exists a unique map σ_2,

$$\sigma_2 : Z \longrightarrow V \otimes W$$

such that $\phi = \sigma_2 \circ \zeta$.
We thus have

$$\begin{aligned} \zeta &= \sigma_1 \circ \sigma_2 \circ \zeta \\ \phi &= \sigma_2 \circ \sigma_1 \circ \phi \end{aligned}$$

and by the uniqueness of the map in the definition of the universal factorization property we obtain

$$\begin{aligned} \sigma_1 \circ \sigma_2 &= \mathbb{I}_Z \\ \sigma_2 \circ \sigma_1 &= \mathbb{I}_{V \otimes W} \end{aligned}$$

so that Z and $V \otimes W$ are isomorphic.

□

[1] To understand this fact consider for example the action of f' on an element of the form $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$. We have

$$\begin{aligned} f'((v_1 + v_2, w) - (v_1, w) - (v_2, w)) &= f'((v_1 + v_2, w)) - f'((v_1, w)) - f'((v_2, w)) \\ &= f(v_1 + v_2, w) - f(v_1, w) - f(v_2, w) \\ &= f(v_1, w) + f(v_2, w) - f(v_1, w) - f(v_2, w) \\ &= 0 \quad , \quad \forall v_1, v_2 \in V, \quad \forall w \in W \quad , \end{aligned} \tag{1.1}$$

where we used the bilinearity of f. With analogous calculations we see that f' vanishes on the other combinations that are used to span $R(V,W)$ so by linearity it vanishes on all $R(V,W)$.

[2] This can be seen by writing the class $v \otimes w$ as $(v,w) + R(V,W)$. But then

$$f'((v,w) + R(V,W)) = f'((v,w)) + f'(R(V,W)) = f'((v,w)) + 0 = f'((v,w))$$

because we remember that $ker(f') \supset R(V,W)$.

Proposition 1.4 (Isomorphism of $V \otimes W$ into $W \otimes V$)
There exists only one isomorphism of $V \otimes W$ onto $W \otimes V$ which $\forall v, w$ sends $v \otimes w$ into $w \otimes v$.

Proof:

Let us consider the universal factorization property of $(V \otimes W, \phi_{VW})$ for $V \times W$ with respect to the map f

$$f : V \times W \longrightarrow W \otimes V$$

defined as $f(v, w) \stackrel{\text{def.}}{=} w \otimes v$. Then we know that there exists only one map f'' such that

$$f'' : V \otimes W \longrightarrow W \otimes V$$

and $f''(v \otimes w) = w \otimes v$.
At the same time we can consider the universal factorization property of $(W \otimes V, \phi_{WV})$ for $W \times V$ with respect to the map g

$$g : W \times V \longrightarrow V \otimes W$$

defined as $g(w, v) \stackrel{\text{def.}}{=} v \otimes w$. Then we know that there exists only one map g'' such that

$$g'' : W \otimes V \longrightarrow V \otimes W$$

and $g''(w \otimes v) = v \otimes w$.
If we pay attention at how the maps f'' and g'' work we have

$$\begin{aligned} f'' \circ g'' &= \mathbb{I}_{W \otimes V} \\ g'' \circ f'' &= \mathbb{I}_{V \otimes W} \end{aligned}$$

so that $W \otimes V$ and $V \otimes W$ are isomorphic.

□

Proposition 1.5 (Isomorphism of $\mathbb{F} \otimes U$ onto U)
Let us consider $\mathbb{F}$ as a 1-dimensional vector space over $\mathbb{F}$. There exists only one isomorphism of $\mathbb{F} \otimes U$ onto U which sends $\rho \otimes u$ into ρu, $\forall \rho \in \mathbb{F}$ and $\forall u \in U$. The same holds for $U \otimes \mathbb{F}$ and U.

Proposition 1.6 (Isomorphism of $(U \otimes V) \otimes W$ onto $U \otimes (V \otimes W)$)
There exists only one isomorphism of $(U \otimes V) \otimes W$ onto $U \otimes (V \otimes W)$ that sends $(u \otimes v) \otimes w$ into $u \otimes (v \otimes w)$, $\forall u \in U$, $\forall v \in V$ and $\forall w \in W$.

We add now some additional observations.

1. The above property implies that it is meaningful to write $U \otimes V \otimes W$ without brackets.

2. By generalizing proposition (1.3) starting from k vector spaces $U_1, \ldots, U_k$ we can define $U_1 \otimes \ldots \otimes U_k$.

3. By generalizing proposition (1.4) to the case of the k-fold tensor product[3] $\forall \pi \in S_k$ there exists only one isomorphism of $U_1 \otimes \ldots \otimes U_k$ onto $U_{\pi(1)} \otimes \ldots \otimes U_{\pi(k)}$ that sends $u_1 \otimes \ldots \otimes u_k$ into $u_{\pi(1)} \otimes \ldots \otimes u_{\pi(k)}$.

4. Without proof we are also going to state the following results:

Proposition 1.7 (Tensor product of functions)
Given vector spaces U_j , V_j, $j = 1, 2$, and given maps

$$f_j : U_j \longrightarrow V_j \quad , \quad j = 1, 2 \quad ,$$

there exists only one map f,

$$f : U_1 \otimes U_2 \longrightarrow V_1 \otimes V_2$$

such that $f(u_1 \otimes u_2) = f(u_1) \otimes f(u_2)$ for all $u_1 \in U_1$ and $u_2 \in U_2$. By definition we will write

$$f \stackrel{\text{def.}}{=} f_1 \otimes f_2 \quad .$$

Proposition 1.8 (Distributive properties of $\otimes$ with respects to $+$.)
Given vector spaces U, V, U_i, V_i, $i = 1, \ldots, k$, the following properties hold:

$$\begin{aligned}(U_1 + \ldots + U_k) \otimes V &= U_1 \otimes V + \ldots + U_k \otimes V \\ U \otimes (V_1 + \ldots + V_k) &= U \otimes V_1 + \ldots + U \otimes V_k.\end{aligned} \tag{1.2}$$

Proposition 1.9 (Basis of tensor product)
Let $\{v_i\}_{i=1,\ldots,m}$ be a basis of V and $\{w_j\}_{j=1,\ldots,n}$ be a basis of W. Then $\{v_i \otimes w_j\}_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ is basis $U \otimes V$. In particular $\dim(U \otimes V) = \dim(U) \dim(V)$.

Proof:

Let U_i be the subspace of U spanned by u_i and V_j the subspace of V spanned by v_j. By proposition (1.8)

$$U \otimes V = \sum_{i=1,\ldots m}^{j=1,\ldots,n} U_i \times V_j.$$

At the same time by proposition (1.5) $U_i \otimes V_j$ is a one dimensional vector space spanned by $u_i \otimes v_j$. This completes the proof.

□

Proposition 1.10 () *Let*

$$L(U^*, V) = \{l : U^* \longrightarrow V, l \quad \text{linear}\}.$$

There exists only one isomorphism,

$$g : U \otimes V \longrightarrow L(U^*, V)$$

[3] S_k is the permutation group of k elements.

Proof:

Let us define a function f,

$$f : U \times V \longrightarrow L(U^*, V) \quad ,$$

such that[4]

$$(f(u,v))(u^*) = u^*(u)v \quad , \quad \forall u \in U \quad , \quad \forall u^* \in U^* \quad , \quad \forall v \in V$$

(remember that $u^*(u) \in \mathbb{F}$). By proposition (1.3) there exists only one g,

$$g : U \otimes V \longrightarrow L(U^*, V)$$

such that $(g(u \otimes v))(u^*) = u^*(u)v$. Let us now fix some basis, $\{u_i\}_{i=1,\dots,m}$ in U, $\{u_i^*\}_{i=1,\dots,m}$ in U^* and $\{v_i\}_{i=1,\dots,n}$ in V. Then $\{g(u_i \otimes v_j)\}_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ is a linearly independent set in $L(U^*, V)$. To show this consider a linear combination of these elements

$$\sum_{i=1,\dots,m}^{j=1,\dots,n} a_{ij} g(u_i \otimes v_j) \quad \text{with} \quad a_{ij} \in \mathbb{F}, \quad \forall i = 1, \dots, m, \quad \forall j = 1, \dots, n,$$

such that

$$\sum_{i=1,\dots,m}^{j=1,\dots,n} a_{ij} g(u_i \otimes v_j) = 0.$$

Then we have that

$$\forall k = 1, \dots m \sum_{i=1,\dots,m}^{j=1,\dots,n} a_{ij} g(u_i \otimes v_j)(u_k^*) = \sum_j a_{kj} v_j = 0$$

which, since the $\{v_i\}_{i=1,\dots,n}$ are linearly independents, implies

$$\forall k = 1, \dots, m, \quad \forall j = 1, \dots, n \quad a_{kj} = 0 \quad .$$

Since the dimensions of $U \otimes V$ and of $L(U^*, V)$ are the same g is an isomorphism and for the definition of the universal mapping property it is also unique.

□

Without proof we also give the additional result:

Proposition 1.11 (Tensor product and duals)
Given vector spaces U and V there exists only one isomorphism g

$$g : U^* \otimes V^* \longrightarrow (U \otimes V)^*$$

such that

$$(g(u^* \otimes v^*))(u \otimes v) = u^*(u)v^*(v), \quad \forall u \in U, \forall u^* \in U^*, \forall v \in V, \forall v^* \in V^*.$$

This result can be generalized to r-fold tensor products.

[4]Remember that $u^* \in U^*$ is an application from U into $\mathbb{F}$. Thus $u^*(u) \in \mathbb{F}$. Moreover f is a function from $U \times V$ into $L(U^*, V)$. Thus $f(u,v)$ is a linear map from U^* into V, i.e. $(f(u,v))(u^*) \in V$.

Notation 1.1 *We set up the following notation:*

$$V_r^s \overset{\text{not.}}{=} \overset{1}{V^*} \times \ldots \times \overset{r}{V^*} \times \overset{1}{V} \times \ldots \times \overset{s}{V}\ .$$

Moreover we set

$$V^s \overset{\text{not.}}{=} V_0^s$$

and

$$V_r \overset{\text{not.}}{=} V_r^0 .$$

Concerning tensor spaces we set

$$T^r(V) \overset{\text{not.}}{=} \overset{1}{V} \otimes \ldots \otimes \overset{r}{V}$$

and

$$T_s(V) \overset{\text{not.}}{=} \overset{1}{V^*} \otimes \ldots \otimes \overset{r}{V^*}\ .$$

Then

$$T_s^r(V) \overset{\text{not.}}{=} T^r(V) \otimes T_s(V)$$

with

$$T_0^0 = \mathbb{F}.$$

Proposition 1.12 (Tensor product and linear mappings)
$T_s(V)$ is isomorphic to the space of s-linear mappings from V^s into $\mathbb{F}$.
$T^r(V)$ is isomorphic to the space of r-linear mappings from V_r into $\mathbb{F}$.
$T_s^r(V)$ is isomorphic to the space of (r,s)-linear mappings from V_r^s into $\mathbb{F}$.

Proof:

We prove only the first result using the generalized result of (1.11). We then see that $T_s(V)$ is the dual vector space of $T^s(V)$. But from the universal factorization property of the tensor product the linear space of mappings of $T^s(V)$ into $\mathbb{F}$ is isomorphic to the space of s-linear mappings of V^s into $\mathbb{F}$. As simple proofs can be given in the other cases.

□

Definition 1.6 (Tensors on V)
We define

$$T_s^r(V) = \{\boldsymbol{T} | \boldsymbol{T} : V_s^r \longrightarrow \mathbb{R},\, \boldsymbol{T} \text{ linear}\}\,,$$

the set of tensors over V.

Proposition 1.13 (Vector space structure of $T_s^r(V)$)
$T_s^r(V)$ is a vector space of dimension n^{r+s} over $\mathbb{R}$.

Proposition 1.14 (Algebra structure of $T_s^r(V)$)
Let V be a vector space of dimension $\dim(V) = n$.

1. *$T^r_s(V)$ together with the operations $(+, \cdot, \otimes)$ (vector space sum, vector space product by a scalar and tensor product) is an algebra over $\mathbb{R}$.*

2. *$\mathcal{B}_{T^r_s}$ defined as*

$$\mathcal{B}_{T^r_s} \stackrel{\text{def.}}{=} \{\boldsymbol{e}_{a_1} \otimes \ldots \otimes \boldsymbol{e}_{a_r} \otimes \boldsymbol{E}_{b_1} \otimes \ldots \otimes \boldsymbol{E}_{b_s} | \\ 1 \leq a_i \leq n, i = 1, \ldots, r, 1 \leq b_j \leq n, j = 1, \ldots, s, \}$$

is a basis of $T^r_s(V)$.

Definition 1.7 (Tensor algebra)
We will call

$$T(V) \stackrel{\text{def.}}{=} \bigoplus_{r,s \leq 0} T^r_s(V)$$

the tensor algebra over V.

Definition 1.8 (Symmetrized tensor)
Let $\boldsymbol{T}$ be an (r, s) tensor, i.e.

$$\boldsymbol{T} = \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_s}}^{1,m} \boldsymbol{T}^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial x_{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \ldots \otimes dx_{j_s}.$$

The symmetrization *of $\boldsymbol{T}$ with respect to the a given subset of vector slots, let us say the k_1-th, ..., k_n-th is the (r, s) tensor $_{(}\boldsymbol{T}_{)}$ defined as*

$$_{(}\boldsymbol{T}_{)}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_r, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_s) = \\ = \frac{1}{n!} \sum_{\sigma \in \mathscr{S}_n} \boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_r, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\sigma(k_1)}, \ldots, \boldsymbol{v}_{\sigma(k_n)}, \ldots, \boldsymbol{v}_s).$$

Definition 1.9 (Antisymmetrized tensor)
The antisymmetrization *of a tensor $\boldsymbol{T}$ with respect to the a given subset of vector slots, let us say the k_1-th, ..., k_n-th is the (r, s) tensor $_{[}\boldsymbol{T}_{]}$ defined as*

$$_{[}\boldsymbol{T}_{]}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_r, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_s) = \\ = \frac{1}{n!} \sum_{\sigma \in \mathscr{S}_n} (-1)^{\sigma} \boldsymbol{T}(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_r, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\sigma(k_1)}, \ldots, \boldsymbol{v}_{\sigma(k_n)}, \ldots, \boldsymbol{v}_s).$$

Similar definitions can be given for 1-form slots, but in general no meaning can be given to symmetrization or antisymmetrization of *mixed* 1-form and vector slots.

1.2.3 Orientation

Let us consider $\Lambda^n(V)$. Since we have $\Lambda^n(V) \cong \mathbb{R}$, then $\Lambda^n(V)/\{0\}$ consists of two connected components.

Definition 1.10 (Orientation on V)
A choice of a connected component of $\Lambda^n(V)/\{0\}$ is an orientation of V.

Proposition 1.15 (Choice of an orientation of V)
The choice of a basis in V is a choice of an orientation on V. This choice is invariant under all endomorphisms of V with positive determinant.

Proof:

If we consider a basis $(v_1, \ldots, v_n)$ of V, then $(v_1^*, \ldots, v_n^*)$ is a basis of V^* (in fact the dual basis) and $v_1^* \wedge \ldots \wedge v_m^*$ is a basis in $\Lambda^n(V)$, i.e. it is in $\Lambda^n(V)/\{0\}$. Thus it selects one of the connected components of $\Lambda^n(V)/\{0\}$, i.e. it defines an orientation on V.

Now consider another basis $(w_1, \ldots, w_n)$ in V. Then $w_i = \sum_j C_{ij} v_j$ and $w_i^* = \sum_j (C^*)_{ij} v_j^*$ with $(C^*)_{ij} = (C^{-1})_{ji}$. Then $w_1^* \wedge \ldots \wedge w_m^* = (\det(C^*))\, v_1^* \wedge \ldots \wedge v_m^*$, where we are interested in the fact that

$$\operatorname{sign}(\det(C^*)) = \operatorname{sign}(\det(C)).$$

C represent an endomorphism of V in the two fixed bases, and if its determinant is positive, then the orientation "chosen" by $(v_1, \ldots, v_n)$ is the same as the one "chosen" by $(w_1, \ldots, w_n)$ since $v_1^* \wedge \ldots \wedge v_m^*$ and $w_1^* \wedge \ldots \wedge w_m^*$ are in the same connected component of $\Lambda^n(V)/\{0\}$.

□

1.2.4 Scalar product

Definition 1.11 (Scalar product)
A real scalar product *over V is a map*

$$\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$$

which is

1. *symmetric, i.e. $\forall \boldsymbol{v}, \boldsymbol{w} \in V$ it satisfies $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle$;*
2. *linear in the first argument, i.e. $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $\lambda, \mu \in \mathbb{R}$ $\Rightarrow \langle \lambda \boldsymbol{u} + \mu \boldsymbol{v}, \boldsymbol{w} \rangle = \lambda \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \mu \langle \boldsymbol{v}, \boldsymbol{w} \rangle$;*
3. *non-degenerate, i.e. such that given $\boldsymbol{v} \in V$, $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0, \forall \boldsymbol{w} \in V \Rightarrow \boldsymbol{v} = \boldsymbol{0}$.*

Given a basis of V if we consider the matrix $g_{ij} = \langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle$ the symmetry assumption implies $g_{ij} = g_{ji}$ and the non-degenerate assumption implies that the matrix g_{ij} is non singular. A scalar product will be called a *metric* on V. When, given a vector $\boldsymbol{v} = \sum_i^{1,n} v^i \boldsymbol{e}_i$, we consider the map

$$\langle \boldsymbol{v}, - \rangle : V \longrightarrow V$$

this is a linear map on V, i.e. $\langle \boldsymbol{v}, - \rangle \in V^* = \Lambda^1(V)$. We can easily determine its components in the dual basis writing

$$\langle \boldsymbol{v}, - \rangle = \sum_j^{1,n} \tilde{v}^j \boldsymbol{E}_j$$

and acting with both sides on $\boldsymbol{w} = \sum_k^{1,n} w^k \boldsymbol{e}_k$:

$$\begin{aligned} & = \quad \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \textstyle\sum_j^{1,n} \tilde{v}_j \boldsymbol{E}^j(\boldsymbol{w}) \quad = \\ \sum_{i,j}^{1,n} g_{ij} v^i w^j & = \qquad\qquad = \sum_j^{1,n} \tilde{v}_j \boldsymbol{E}_j \left(\sum_k^{1,n} w^k \boldsymbol{e}_k \right) \\ \sum_{i,j}^{1,n} g_{ij} v^i w^j & = \qquad\qquad = \sum_{j,k}^{1,n} \tilde{v}_j w^k \boldsymbol{E}_j \left(\boldsymbol{e}_k \right) \\ \sum_j^{1,n} \left(\sum_i^{1,n} g_{ij} v^i \right) w^j & = \qquad\qquad = \sum_j^{1,n} \tilde{v}_j w^j. \end{aligned} \tag{1.3}$$

Thus

$$\tilde{v}_j = \sum_i^{1,n} g_{ij} v^i.$$

The converse is also true: if we have a 1-form $\boldsymbol{\omega} = \sum_i^{1,n} \omega_i \boldsymbol{E}^i \in V^*$ we can associate to it a unique vector $\boldsymbol{w} \in V$, whose components are defined as $w^i = \sum_j^{1,n} (g^{-1})_{ij} \omega_j$. Thus the metric induces a natural isomorphisms between V and V^*. Since the action of an $\boldsymbol{\omega} \in V^*$ is independent from the definition of a metric on V, we will keep the notation $\boldsymbol{\omega}(\boldsymbol{v})$ and we will not rewrite it in terms of the scalar product.

Definition 1.12 (Signature and Lorentzian metric)
Let $\langle -, - \rangle$ be a metric on V. The signature *of the metric is the number of positive eigenvalues of the matrix g_{ij} minus the number of negative eigenvalues. A metric of signature $m-2$ is called a* Lorentzian metric*.*

Definition 1.13 (Timelike, spacelike and null vectors)
Let $\langle -, - \rangle$ be a Lorentzian metric on the vector space V. A vector $\boldsymbol{v} \in V$ is timelike *if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0$,* spacelike *if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$ and* null *if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$.*

1.3 Topology preliminaries

Definition 1.14 (Topology and open sets)
Let $\mathscr{S}$ be a set and $\mathcal{T}$ a collection of subsets of $\mathscr{S}$ such that:

1. *$\mathscr{S} \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$;*
2. *given $n \in \mathbb{N}$, $A_i \in \mathcal{T}$, $i = 1, \ldots, n \Rightarrow \bigcap_i^{1,n} A_i \in \mathcal{T}$;*
3. *given a collection $\{A_n\}_{n \in \mathbb{N}}$, $A_n \in \mathcal{T}$ $\forall n \in \mathbb{N} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{T}$.*

$\mathcal{T}$ is called a topology *on $\mathscr{S}$; its elements are called* open sets*.*

Definition 1.15 (Topological space)
Let $\mathscr{S}$ be a set and $\mathcal{T}$ a topology on $\mathscr{S}$. The couple $(\mathscr{S}, \mathcal{T})$ is a topological space.

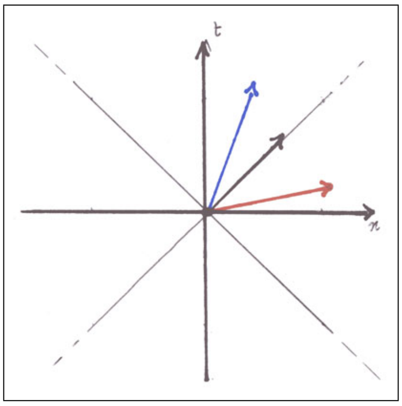

Figure 1.1: Timelike, spacelike and null vectors.

Definition 1.16 (Neighborhood)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space and $\mathsf{p} \in \mathscr{S}$. A neighborhood *of p is an open set $P \in \mathcal{T}$ such that $\mathsf{p} \in P$.*

Definition 1.17 (Cover)
Let $\mathscr{S}$ be a set and $\mathscr{U} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ a collection of subsets of $\mathscr{S}$ indexed by a set $\mathcal{A}$. $\mathscr{U}$ is called a cover *of $\mathscr{S}$ if $\bigcup_{\alpha \in \mathcal{A}} S_\alpha = \mathscr{S}$.*

Definition 1.18 (Subcover)
Let $\mathscr{S}$ be a set and $\mathscr{U} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ a cover of $\mathscr{S}$. Let $\mathcal{A}' \subseteq \mathcal{A}$. Then $\mathscr{U}' = \{S_{\alpha'}\}_{\alpha' \in \mathcal{A}'}$ is a subcover *of the cover $\mathscr{U}$ of $\mathscr{S}$.*

Of course, a subcover is itself a cover.

Definition 1.19 (Refinement)
Let $\mathscr{S}$ be a set and $\mathscr{U} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ a cover of $\mathscr{S}$. Another cover $\mathscr{V} = \{S'_\beta\}_{\beta \in \mathcal{B}}$ of $\mathscr{S}$ is called a refinement *of $\mathscr{U}$ if $\forall \beta \in \mathcal{B}$, $\exists \alpha \in \mathcal{A}$ such that $S'_\beta \subset S_\alpha$.*

Definition 1.20 (Open cover)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space and let $\mathcal{O} = \{O_\alpha\}_{\alpha \in \mathcal{A}}$ be a cover of $\mathscr{S}$. $\mathcal{O}$ is open cover *of $\mathscr{S}$ if $S_\alpha \in \mathcal{T}$ $\forall \alpha \in \mathcal{A}$.*

Definition 1.21 (Locally finite open cover)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space and $\mathcal{O} = \{O_\alpha\}_{\alpha \in \mathcal{A}}$ an open cover of $\mathscr{S}$. $\mathcal{O}$ is a locally finite open cover *of $\mathscr{S}$ if $\forall \mathsf{s} \in \mathscr{S}$ there exists W open neighborhood of s such that $\{O_i | O_i \cap W \neq \emptyset\}$ is a finite set.*

Definition 1.22 (Compact topological space)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space. $\mathscr{S}$ is compact *if every open cover of $\mathscr{S}$ admits a finite subcover.*

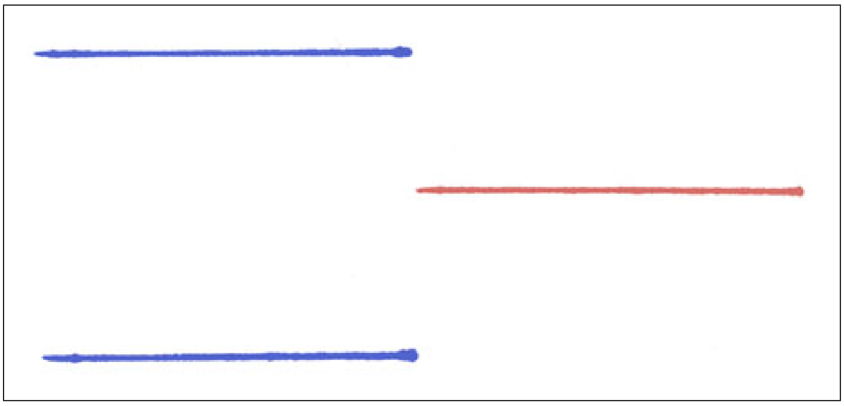

Figure 1.2: Typical example of a non-Hausdorff topological space.

Definition 1.23 (Paracompact topological space)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space. $\mathscr{S}$ is paracompact *if every open cover of $\mathscr{S}$ admits a locally finite open refinement.*

Definition 1.24 (Hausdorff topological space)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space. $\mathscr{S}$ is a Hausdorff space *if $\forall \mathfrak{p}, \mathfrak{q} \in \mathscr{S}$ there exist P and Q, open neighborhoods of $\mathfrak{p}$ and $\mathfrak{q}$ respectively, such that $P \cap Q = \emptyset$.*