Chapter 5

Relativity in Einstein's Words

"Ok, see you tomorrow, 6 p.m. o'clock, at the restaurant". Just a simple sentence, certainly with a clear meaning for the persons speaking and hearing. But what if we reflect carefully about how well defined are the space and time determination that we find in it? Has that sentence a precise meaning for everyone? In all fairness we have to answer *no*! In all fairness that sentence perhaps will have a precise meaning only for the two persons speaking: they have the required common background knowledge to make the very vague space and time determinations in the sentence, very precise for them. At least they need a common background knowledge, which is not explicit in the sentence, if they are going to meet, not by change, on the next day. For what we are interested in, they have some common assumptions on the meaning of the space and time determinations, that make them meaningfull to them and useful.

According to a similar perspective we can understand the meaning attributed to the concepts of space and time before the development of *special relativity*. It is worth to emphasize that the mathematical tools underlying the theory were already known since years. What was missing was a deep reflection about the concepts of space and time.

5.1 Reflections on spacetime

5.1.1 Space and time in pre-relativistic physics

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5.1.2 Classical mechanics and its framework

The pre-relativistic concepts of space and time have been developed together with one branch of science, namely mechanics. As Einstein itself says, mechanics is usually defined as having the scope of *defining as bodies change their position in space with time*, but the above definition need to be carefully discussed. Especially we need a characterization of the ideas of *position*, *space* and *time*.

Position and space

Since the recognition of Galilean invariance to the idea of *motion in space* no precise meaning is associated. Instead we are used to speak about *motion with respect to a given reference system (rigid body).* Already Berkley observed that the concept of absolute motion cannot be conceived: motion is relative, which means we always have to ask ourselves, relatively to what we are describing motion. Thus position and trajectories are necessarily defined in connection with a given reference system. If we want to free ourselves from the linkage with a *ones and for all* chosen reference system it is important to know how to *translate* our description of relative motion when we refer it to a different reference system.

The principle of special relativity

In classical mechanics we acknowledge that some particular reference systems can be singled out, the class of *inertial systems*: these systems have the common property of being in uniform translational motion one with respect to the other. In these systems it is easy to formulate the classical laws of mechanics, since the *group* of transformations that connect these reference systems, the *Galilei group* leave Newton's equations (i.e. the equations of motions) invariant. In first Newton's law, *a body sufficiently far away from other bodies keeps its state of no motion or of uniform translational motion* there is not only a characterization of motion, but *mainly* of the class of reference systems that allow a description of motion in the terms given by classical mechanics, i.e. in which the laws of Newtonian mechanics are valid. The fact that *if some natural phenomena respect some general laws in a given reference system, then they respect the same laws in a reference system which is uniformly translating with respect to the first one* is called the **principle of relativity in the restricted sense**.

The law of addition of velocities

In the framework of classical mechanics we obtain the following law of composition of velocities.

Proposition 5.1 (Galilean law of composition of velocities)
Let K be a Galilean reference system and $\tilde{K}$ another Galilean reference systems in motion with uniform velocity $\boldsymbol{V}$ with respect to K. If a body has a speed $\boldsymbol{v}$ with respect to K and speed $\tilde{\boldsymbol{v}}$ with respect to $\tilde{K}$ then we have $\tilde{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{V}$.

5.1.3 Electrodynamics

The law of propagation of light

What we have shortly discussed above is quite well verified experimentally in the domain of classical mechanics. Unfortunately there is a phenomenon that creates a crucial problem in the above considered framework: it is the *experimentally as well verified fact* that *light propagates in vacuum rectilinearly with constant velocity c*. Einstein calls this the **law of propagation of light**. systems. ... Many experiments have been conducted to test this property, and in the limits of measurement uncertainties no violation of this law has been found. Nevertheless this law is in contrast with the principle of relativity in the restricted sense as explained above.

5.1.4 The consistency problem

Now the above discussion quickly leads to an inconsistency.

1. Let us assume that the principle of relativity holds in the context of classical Mechanics: then the Galilean law of the composition of velocities also holds.

2. Let us then consider a light ray, that we experimentally measure as propagating in vacuum with velocity c in the direction of the x axis of a given reference system K.

3. Let us consider another reference system $\tilde{K}$ oriented as K and translating uniformly with velocity V in the direction of the x axis of K (which is the $\tilde{x}$ axis of $\tilde{K}$).

4. With respect to $\tilde{K}$ the light ray has a velocity directed as the $\tilde{x}$ axis which is $\tilde{c} = c + V$, since we use a particularly simple case of the law of the composition of velocity. **But** according to the principle of relativity, the law of propagation of light should hold also in $\tilde{K}$, i.e. we should have $\tilde{c} = c$, which is of course impossible for $V \neq 0$.

We *apparently* face the dichotomic choice of denying or the law of propagation of light in vacuum or the principle of relativity.

5.2 Einstein solution: a reflection about time

Einstein's original idea about the above presented problem starts with a reflection on it and on the exact terms in which it is posed. In particular Einstein calls the contradiction between the principle of relativity and the law of propagation of light an *apparent contradiction*. It stems from the fact that both, the principle of relativity and the law of propagation of light, are quite well verified experimentally. We have seen they conflict according to the classical law of composition of velocity, of which we assume the validity in the framework of our knowledge of classical mechanics. This result, as well as many others, are nevertheless based on our assumptions about the structure of space and time. In particular Einstein re-analyzes these concepts, particularly the *time* concepts, exactly on the basis of the two well verified and apparently contradicting principles we have discussed above.

5.2.1 Simultaneity

Einstein's reasoning starts from the idea of simultaneity. How can we define this concept? To obtain an operative definition let us imagine we want to define a procedure to *decide* if two lighting hit two far away places A and B simultaneously[1].

[1] We give for granted that simultaneity at the same place can be easily defined by an observer according to an intuitive idea

> '[...]After thinking the matter over for some time you then offer the following suggestion with which to test simultaneity. By measuring along the rails, the connecting line AB should be measured up and an observer placed at the mid-point M of the distance AB. This observer should be supplied with an arrangement (e.g. two mirrors inclined at 90°) which allows him visually to observe both places A and B at the same time. If the observer perceives the two flashes of lighting at the same time, then they are simultaneous.[...]'

Their is an objection we can rise to this definition. It seems that to know that the two events at A and B are simultaneously we need to know that light moves on the two halves of the segment with the same speed. But to measure this speed we already need a way to measure time, so we are in a vicious circle. To fight this objection Einstein himself writes:

> '[...]After further consideration you cast a somewhat disdainful glance at me –and rightly so– and you declare:
>
> > "I maintain my previous definition nevertheless, because in reality it assumes absolutely nothing about light. There is only one demand to be made of the definition of simultaneity, namely, that in every real case it must supply us with an empirical decision as to whether or not the conception that has to be defined is fulfilled. That my definition satisfies this demand is indisputable. That light requires the same time to transverse the path $A \to M$ as for the path $B \to M$ is in reality neither a *supposition nor a hypothesis* about the physical nature of light, but a stipulation which I can make of my own freewill to arrive at a definition of simultaneity."[...]'

From the above quotation we see the original point in Einstein's thoughts. Simultaneity at different places is not something on which we just agree according to a universally accepted definition of universal time (as in classical physics). It is a concept that we define operationally according to a well defined procedure. These procedure gives us the possibility of synchronizing clocks at different places by sending light signals from one place to the other. Note the peculiar role that light (more generally electromagnetic phenomena) have in this definition. Nevertheless we want to stress out that about the speed of light we have nothing more than a *stipulation* that we make to *arrive at a definition of simultaneity*. What are the consequences of this definitions on some everyday concepts like simultaneity and spatial distance? We are going to answer to this question in what follows.

5.2.2 Relativity of simultaneity

5.2.3 Relativity of distance

5.3 Special Relativity

5.4 Lorentz transformations

5.4.1 The algebraic derivation

Let us consider two reference systems K and $\tilde{K}$. Let the coordinates on K be x, y, z and let t be the time shown by a clock in the origin O of K. Correspondingly let the coordinates on $\tilde{K}$ be $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and let $\tilde{t}$ be the time shown by a clock in the origin $\tilde{O}$ of $\tilde{K}$. Moreover let K and $\tilde{K}$ be such that corresponding axes are parallel (with the x and $\tilde{x}$ axes collinear) and such that $\tilde{K}$ is moving with velocity V with respect to K in the common direction of the axes x, $\tilde{x}$. Every event is represented in K by the four numbers (x, y, z, t) and in $\tilde{K}$ by *other* four numbers $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$. To begin let us restrict our attention to the x, $\tilde{x}$ and t, $\tilde{t}$ coordinates. A light ray propagating along the x axis in K satisfies the law of propagation of light, i.e. it can be described by the equation

$$x = ct \quad \Rightarrow \quad x - ct = 0, \tag{5.1}$$

where c is the speed of light. Because of the relativity principle the law of propagation of light at the same speed c also holds in $\tilde{K}$, where the propagation of the light ray will be defined by the equation

$$\tilde{x} - c\tilde{t} = 0. \tag{5.2}$$

All events satisfying (5.1) must satisfy also (5.2), which is the case if

$$\left(\tilde{x} - c\tilde{t}\right) = \lambda_1 \left(x - ct\right). \tag{5.3}$$

Let us now apply the same considerations to a light ray propagating along the negative x or $\tilde{x}$ direction. We obtain

$$\left(\tilde{x} + c\tilde{t}\right) = \lambda_2 \left(x + ct\right). \tag{5.4}$$

Let us sum the two equations above, after setting

$$\alpha_1 = \frac{\lambda_1 + \lambda_2}{2} \quad , \quad \alpha_2 = \frac{\lambda_1 + \lambda_2}{2},$$

to obtain

$$\begin{cases} \tilde{x} &= -\alpha_2 ct + \alpha_1 x \\ c\tilde{t} &= \alpha_1 ct - \alpha_2 x \end{cases} . \tag{5.5}$$

We have to determine now the constants α_1 and α_2. Let us consider the origin $\tilde{O}$ of $\tilde{K}$. $\tilde{O}$ is identified by $\tilde{x} = 0$ so that from the first equation in (5.5) we obtain

$$x = \frac{\alpha_2}{\alpha_1} ct.$$

But x/t is the velocity of the reference system $\tilde{K}$ as seen by K, which is V, so that from the above

$$\tilde{O}: \quad \frac{x}{ct} = \frac{V}{c} = \frac{\alpha_2}{\alpha_1}.$$

Let us now consider a rod of length ΔL. Applying the principle of relativity to the measurement of the length of the rod, we know that its length when at rest in K as seen by $\tilde{K}$ must be the same as its length when at rest in $\tilde{K}$ as seen by K. Let us call these two lengths $\Delta L_{\tilde{K}}$ and ΔL_K respectively.

$\Delta L_{\tilde{K}}$) Let us put the rod at rest in K with one end in $x_1 = 0$ and the other in $x_2 = \Delta L$ at the time $t = 0$ of K. Transforming to $\tilde{K}$ using the first of equations (5.5) we obtain as the start and end positions of the rods

$$\tilde{x}_1 = 0 \quad , \quad \tilde{x}_2 = \alpha_1 \Delta l,$$

so that

$$\Delta L_{\tilde{K}} = \tilde{x}_2 - \tilde{x}_1 = \alpha_1 \Delta l.$$

ΔL_K) Let us put the rod at rest in $\tilde{K}$ at $\tilde{t} = 0$ with one end in $\tilde{x}_1 = 0$ and the other in $\tilde{x}_2 = \Delta L$. Using equations (5.5) we obtain for the first end

$$\begin{cases} 0 &= -\alpha_2 ct + \alpha_1 x_1 \\ 0 &= \alpha_1 ct - \alpha_2 x_1 \end{cases}$$

or, which is the same,

$$\begin{cases} 0 &= -\alpha_2 ct + \alpha_1 x_1 \\ ct &= \alpha_2 x_1 / \alpha_1 \end{cases}$$

i.e. $x_1 = 0$. For the second end we have

$$\begin{cases} \Delta l &= -\alpha_2 ct + \alpha_1 x_1 \\ 0 &= \alpha_1 ct - \alpha_2 x_1 \end{cases}$$

and again, obtaining ct from the second equation and substituting into the first,

$$\begin{cases} \Delta l &= -\alpha_2^2 x_1 / \alpha_1 + \alpha_1 x_1 \\ ct &= \alpha_2 x_1 / \alpha_1 \end{cases}$$

so that the two end points in K are

$$x_1 = 0 \quad , \quad x_2 = \frac{\Delta l}{\alpha_1} \frac{1}{\left(1 - \frac{\alpha_2^2}{\alpha_1^2}\right)}$$

and we obtain

$$\Delta L_K = \frac{\Delta l}{\alpha_1} \frac{1}{\left(1 - \frac{\alpha_2^2}{\alpha_1^2}\right)}.$$

As we said above, because of the relativity principle we have

$$\Delta L_{\tilde{K}} = \Delta L_K,$$

i.e.

$$\alpha_1 \Delta l = \frac{\Delta l}{\alpha_1} \frac{1}{\left(1 - \frac{\alpha_2^2}{\alpha_1^2}\right)}.$$

We conclude

$$\alpha_1 = \left(1 - \frac{\alpha_2^2}{\alpha_1^2}\right)^{-\frac{1}{2}}.$$

We have already determined the ratio of α_1 and α_2 from which we see that

$$\alpha_1 = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}} \stackrel{\text{def.}}{=} \gamma.$$

Then

$$\alpha_2 = \frac{V/c}{\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}}} \stackrel{\text{def.}}{=} \gamma\beta$$

and the transformation equations can be written as

$$\begin{cases} c\tilde{t} &= \gamma(ct - \beta x) \\ \tilde{x} &= \gamma(-\beta ct + x) \end{cases}. \quad (5.6)$$

These transformation laws are called Lorentz transformations. We observe that the quantity

$$x^2 - c^2t^2 \stackrel{\text{def.}}{=} s^2$$

is left invariant by the Lorentz transformations. If we extend this transformations to the full system of coordinates we have

$$\begin{cases} c\tilde{t} &= \gamma(ct - \beta x) \\ \tilde{x} &= \gamma(-\beta ct + x) \\ \tilde{y} &= y \\ \tilde{z} &= z \end{cases} \quad (5.7)$$

from which we see that axes orthogonal to the relative motion direction are not affected by the transformation.

5.4.2 Space and Time in relativistic physics

The above picture of space time we gave in pre-relativistic physics was quite implicit in the mind of scientists. As in our naive example at the beginning of this section, there where no doubts that the Newtonian structure of space and time was quite a proper framework for the description of physical phenomena. We must admit that many success in science and technology could be effectively achieved inside the above framework. Nevertheless the pre-relativistic ideas about space and time where not connected to an operational idea (we are not speaking of the units of measurement of space and time, which were defined according to some world-wide accepted operative definitions, but of the real essence of the concepts of space and time themselves). The deepest meaning of special relativity can be traced back to a re-interpretation of these concepts according to an operational point of view. This is a great advantage, not only because these gave us the opportunity to develop a completely new and important framework for the description of physical phenomena, but also because ... clearly stated. This is a basic aspect that can help us in further developing our understanding of the concepts of space and time.