Chapter 4

Exercises

4.1 Connection and Covariant Derivative

Problem 4.1 (Transformation law of $\Gamma^{\alpha}_{\mu\nu}$)
Let us consider a $(1,2)$ tensor $T^{\alpha}_{\mu\nu}$. What is its transformation law? How do the connection coefficients $\Gamma^{\alpha}_{\mu\nu}$ transform under a change of coordinates? Are the $\Gamma^{\alpha}_{\mu\nu}$ a tensor?

Solution:

Let us fix a basis $\{\boldsymbol{e}_\mu\}_{\mu=0,\ldots,3}$ in the tangent space and let $\{\boldsymbol{E}^\mu\}_{\mu=0,\ldots,3}$ be the dual basis. Then let us consider a change of coordinates defined by

$$\boldsymbol{e}'_\mu = \Lambda_\mu{}^\nu \boldsymbol{e}_\nu,$$

so that on the dual basis

$$\boldsymbol{E}'^\mu = (\Lambda^{-1})_\nu{}^\mu \boldsymbol{E}^\nu,$$

where $\{\boldsymbol{e}'_\mu\}_{\mu=0,\ldots,3}$ is the new basis and $\{\boldsymbol{E}'^\mu\}_{\mu=0,\ldots,3}$ the corresponding dual. The components of a $(1,2)$ tensor in a given basis, let us say $\{\boldsymbol{e}_\mu\}_{\mu=0,\ldots,3}$, are given by

$$T^\lambda_{\mu\nu} = \boldsymbol{T}(\boldsymbol{E}^\lambda, \boldsymbol{e}_\mu, \boldsymbol{e}_\nu).$$

On the other hand we have

$$\begin{aligned}
T'^{\alpha}_{\beta\gamma} &= \boldsymbol{T}(\boldsymbol{E}'^\alpha, \boldsymbol{e}'_\beta, \boldsymbol{e}'_\gamma) \\
&= \boldsymbol{T}((\Lambda^{-1})_\lambda{}^\alpha \boldsymbol{E}^\lambda, \Lambda_\beta{}^\mu \boldsymbol{e}_\mu, \Lambda_\gamma{}^\nu \boldsymbol{e}_\nu) \\
&= (\Lambda^{-1})_\lambda{}^\alpha \Lambda_\beta{}^\mu \Lambda_\gamma{}^\nu T(\boldsymbol{E}^\lambda, \boldsymbol{e}_\mu, \boldsymbol{e}_\nu) \\
&= (\Lambda^{-1})_\lambda{}^\alpha \Lambda_\beta{}^\mu \Lambda_\gamma{}^\nu T^\lambda_{\mu\nu}.
\end{aligned}$$

By comparison of the first and last lines we get

$$T'^{\alpha}_{\beta\gamma} = (\Lambda^{-1})_\lambda{}^\alpha \Lambda_\beta{}^\mu \Lambda_\gamma{}^\nu T^\lambda_{\mu\nu}.$$

Let us now apply a similar procedure to the connection. We do not want to be restricted to a coordinate basis, so we start from the definition of the Γ's[1],

$$
\begin{aligned}
{\Gamma'}^{\lambda}_{\mu\nu} &= \boldsymbol{E}'^{\lambda}(D(\boldsymbol{e}'_{\mu}, \boldsymbol{e}'_{\nu})) \\
&= (\Lambda^{-1})_{\alpha}{}^{\lambda} \boldsymbol{E}^{\alpha}(D(\Lambda_{\mu}{}^{\beta}\boldsymbol{e}_{\beta}, \Lambda_{\nu}{}^{\gamma}\boldsymbol{e}_{\gamma})) \\
&= (\Lambda^{-1})_{\alpha}{}^{\lambda} \boldsymbol{E}^{\alpha}(\Lambda_{\mu}{}^{\beta} D(\boldsymbol{e}_{\beta}, \Lambda_{\nu}{}^{\gamma}\boldsymbol{e}_{\gamma})) \\
&= (\Lambda^{-1})_{\alpha}{}^{\lambda} \boldsymbol{E}^{\alpha}(\Lambda_{\mu}{}^{\beta} D(\boldsymbol{e}_{\beta}, \Lambda_{\nu}{}^{\gamma}\boldsymbol{e}_{\gamma})) \\
&= (\Lambda^{-1})_{\alpha}{}^{\lambda} \boldsymbol{E}^{\alpha}(\Lambda_{\mu}{}^{\beta}\Lambda_{\nu}{}^{\gamma} D(\boldsymbol{e}_{\beta}, \boldsymbol{e}_{\gamma}) + \Lambda_{\mu}{}^{\beta}\boldsymbol{e}_{\beta}(\Lambda_{\nu}{}^{\gamma})\boldsymbol{e}_{\gamma}) \\
&= (\Lambda^{-1})_{\alpha}{}^{\lambda} \left[\boldsymbol{E}^{\alpha}(\Lambda_{\mu}{}^{\beta}\Lambda_{\nu}{}^{\gamma} D(\boldsymbol{e}_{\beta}, \boldsymbol{e}_{\gamma})) + \boldsymbol{E}^{\alpha}(\Lambda_{\mu}{}^{\beta}\boldsymbol{e}_{\beta}(\Lambda_{\nu}{}^{\gamma})\boldsymbol{e}_{\gamma})\right] \\
&= (\Lambda^{-1})_{\alpha}{}^{\lambda} \left[\Lambda_{\mu}{}^{\beta}\Lambda_{\nu}{}^{\gamma} \boldsymbol{E}^{\alpha}(D(\boldsymbol{e}_{\beta}, \boldsymbol{e}_{\gamma})) + \Lambda_{\mu}{}^{\beta}\boldsymbol{e}_{\beta}(\Lambda_{\nu}{}^{\gamma})\boldsymbol{E}^{\alpha}(\boldsymbol{e}_{\gamma})\right] \\
&= (\Lambda^{-1})_{\alpha}{}^{\lambda}\Lambda_{\mu}{}^{\beta}\Lambda_{\nu}{}^{\gamma}\Gamma^{\alpha}_{\beta\gamma} + (\Lambda^{-1})_{\alpha}{}^{\lambda}\Lambda_{\mu}{}^{\beta}\boldsymbol{e}_{\beta}(\Lambda_{\nu}{}^{\gamma})\delta^{\alpha}_{\gamma} \\
&= (\Lambda^{-1})_{\alpha}{}^{\lambda}\Lambda_{\mu}{}^{\beta}\Lambda_{\nu}{}^{\gamma}\Gamma^{\alpha}_{\beta\gamma} + (\Lambda^{-1})_{\alpha}{}^{\lambda}\Lambda_{\mu}{}^{\beta}\boldsymbol{e}_{\beta}(\Lambda_{\nu}{}^{\alpha}).
\end{aligned}
$$

Thus

$$
{\Gamma'}^{\lambda}_{\mu\nu} = (\Lambda^{-1})_{\alpha}{}^{\lambda}\Lambda_{\mu}{}^{\beta}\Lambda_{\nu}{}^{\gamma}\Gamma^{\alpha}_{\beta\gamma} + (\Lambda^{-1})_{\alpha}{}^{\lambda}\Lambda_{\mu}{}^{\beta}\boldsymbol{e}_{\beta}(\Lambda_{\nu}{}^{\alpha})
$$

and, because of the last term, the connection symbols are not the component of a tensor.

□

Problem 4.2 (Compatibility condition in coordinates)
Let us consider the covariant derivative associated with the only symmetric connection compatible with a given metric on a manifold $(\mathscr{M}, \mathcal{F})$. Prove that this implies that the metric is covariantly constant, i.e. $\nabla_{\mu} g_{\alpha\beta} = 0$, and that this also implies $\nabla_{\mu} g^{\alpha\beta} = 0$.

Solution:

We first compute $\boldsymbol{e}_{\tau}(\delta^{\nu}_{\mu})$:

$$
\begin{aligned}
0 &= \boldsymbol{e}_{\tau}(\delta^{\nu}_{\mu}) \\
&= \boldsymbol{e}_{\tau}(g_{\mu\alpha}g^{\alpha\nu}) \\
&= g_{\mu\alpha}\boldsymbol{e}_{\tau}(g^{\alpha\nu}) + \boldsymbol{e}_{\tau}(g_{\mu\alpha})g^{\alpha\nu} \\
&\Rightarrow g_{\mu\alpha}\boldsymbol{e}_{\tau}(g^{\alpha\nu}) = -g^{\alpha\nu}\boldsymbol{e}_{\tau}(g_{\mu\alpha}) \\
&\Rightarrow g^{\beta\mu}g_{\mu\alpha}\boldsymbol{e}_{\tau}(g^{\alpha\nu}) = -g^{\beta\mu}g^{\alpha\nu}\boldsymbol{e}_{\tau}(g_{\mu\alpha}) \\
&\Rightarrow \delta^{\beta}_{\alpha}\boldsymbol{e}_{\tau}(g^{\alpha\nu}) = -g^{\beta\mu}g^{\alpha\nu}\boldsymbol{e}_{\tau}(g_{\mu\alpha})
\end{aligned}
$$

so that renaming indices in a convenient way:

$$
\boldsymbol{e}_{\tau}(g^{\mu\nu}) = -g^{\mu\alpha}g^{\nu\beta}\boldsymbol{e}_{\tau}(g_{\alpha\beta}).
$$

Now we turn to establish the main result:

$$
\begin{aligned}
D(\boldsymbol{e}_{\gamma}, \boldsymbol{g}) &= D(\boldsymbol{e}_{\gamma}, g^{\mu\nu}\boldsymbol{e}_{\mu}\otimes\boldsymbol{e}_{\nu}) \\
&= \boldsymbol{e}_{\gamma}(g^{\mu\nu})\boldsymbol{e}_{\mu}\otimes\boldsymbol{e}_{\nu} + g^{\mu\nu}D(\boldsymbol{e}_{\gamma}, \boldsymbol{e}_{\mu}\otimes\boldsymbol{e}_{\nu})
\end{aligned}
$$

[1] We remember that the covariant derivative of a vector in a given direction is a vector again, whose components are expressed in terms of the Γ's. On the other hand the component of a vector in the direction of the basis vector $\boldsymbol{e}_{\lambda}$ can be found applying the 1-form $\boldsymbol{E}^{\lambda}$ to the vector itself.

$$\begin{aligned}
&= -g^{\mu\alpha}g^{\nu\beta}\boldsymbol{e}_\gamma(g_{\alpha\beta})\boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu + \\
&\quad +g^{\mu\nu}\boldsymbol{e}_\mu \otimes D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\nu) + g^{\mu\nu}D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\mu) \otimes \boldsymbol{e}_\nu \\
&= -g^{\mu\alpha}g^{\nu\beta}\boldsymbol{e}_\gamma(\langle \boldsymbol{e}_\alpha, \boldsymbol{e}_\beta\rangle)\boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu + \\
&\quad +\boldsymbol{E}^\nu \otimes D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\nu) + D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\mu) \otimes \boldsymbol{E}^\mu \\
&= -g^{\mu\alpha}g^{\nu\beta}\left[\langle D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\alpha), \boldsymbol{e}_\beta\rangle + \langle D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\beta), \boldsymbol{e}_\alpha\rangle\right]\boldsymbol{e}_\mu \otimes \boldsymbol{e}_\nu + \\
&\quad +\boldsymbol{E}^\nu \otimes \boldsymbol{E}^\beta \langle D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\nu), \boldsymbol{e}_\beta\rangle + \langle D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\mu), \boldsymbol{e}_\beta\rangle \boldsymbol{E}^\beta \otimes \boldsymbol{E}^\mu \\
&= -\langle D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\alpha), \boldsymbol{e}_\beta\rangle \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta - \langle D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\beta), \boldsymbol{e}_\alpha\rangle \boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta + \\
&\quad +\langle D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\nu), \boldsymbol{e}_\beta\rangle \boldsymbol{E}^\nu \otimes \boldsymbol{E}^\beta + \langle D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\mu), \boldsymbol{e}_\beta\rangle \boldsymbol{E}^\beta \otimes \boldsymbol{E}^\mu \\
&= 0.
\end{aligned}$$

So we have shown that the metric tensor is covariantly constant, and thus $\nabla_\mu g_{\alpha\beta} = \nabla_\mu g^{\alpha\beta} = 0$, since we can write

$$\begin{aligned}
0 &= D(\boldsymbol{e}_\mu, \boldsymbol{g}) \\
&= (\nabla_\mu \boldsymbol{g})_{\alpha\beta}\boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta \\
&= (\nabla_\mu \boldsymbol{g})^{\alpha\beta}\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta \\
&= (\nabla_\mu g_{\alpha\beta})\boldsymbol{E}^\alpha \otimes \boldsymbol{E}^\beta \\
&= (\nabla_\mu g^{\alpha\beta})\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta
\end{aligned}$$

As a consequence of this result we can also obtain the following:

$$\begin{aligned}
0 &= D(\boldsymbol{e}_\gamma, \boldsymbol{g}) \\
&= D(\boldsymbol{e}_\gamma, g_{\mu\nu}\boldsymbol{E}^\mu \otimes \boldsymbol{E}^\nu) \\
&= D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\nu \otimes \boldsymbol{E}^\nu) \\
&= D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\nu) \otimes \boldsymbol{E}^\nu + \boldsymbol{e}_\nu \otimes D(\boldsymbol{e}_\gamma, \boldsymbol{E}^\nu) \\
&= \boldsymbol{E}^\mu(D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\nu))\boldsymbol{e}_\mu \otimes \boldsymbol{E}^\nu + \boldsymbol{e}_\nu \otimes \boldsymbol{E}^\rho \langle \boldsymbol{e}_\rho, D(\boldsymbol{e}_\gamma, \boldsymbol{E}^\nu)\rangle \\
&\Rightarrow \boldsymbol{E}^\mu(D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\nu))\boldsymbol{e}_\mu \otimes \boldsymbol{E}^\nu = -\langle \boldsymbol{e}_\nu, D(\boldsymbol{e}_\gamma, \boldsymbol{E}^\mu)\rangle \boldsymbol{e}_\mu \otimes \boldsymbol{E}^\nu \\
&\Rightarrow \boldsymbol{E}^\mu(D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\nu))\boldsymbol{E}^\nu = -\langle \boldsymbol{e}_\nu, D(\boldsymbol{e}_\gamma, \boldsymbol{E}^\mu)\rangle \boldsymbol{E}^\nu \\
&\Rightarrow D(\boldsymbol{e}_\gamma, \boldsymbol{E}^\mu) = -\boldsymbol{E}^\mu(D(\boldsymbol{e}_\gamma, \boldsymbol{e}_\nu))\boldsymbol{E}^\nu, \qquad (4.1)
\end{aligned}$$

i.e. the covariant derivative of a 1-form.

□

Problem 4.3 (Useful identities)
Prove that the following identities are satisfied (it could be easier to prove some of them using one or more of the previously *established identities).*

1. $\partial_\alpha g_{\mu\nu} = \Gamma_{\mu\nu\alpha} + \Gamma_{\nu\mu\alpha}$;
2. $g_{\mu\sigma}\partial_\tau g^{\sigma\nu} = -(\partial_\tau g_{\mu\sigma})g^{\sigma\nu}$;
3. $\partial_\nu g^{\alpha\beta} + \Gamma^\alpha_{\mu\nu}g^{\nu\beta} + \Gamma^\beta_{\mu\nu}g^{\nu\alpha} = 0$;
4. $\partial_\alpha g = -g g_{\mu\nu}\partial_\alpha g^{\mu\nu} = g g^{\mu\nu}\partial_\alpha g_{\mu\nu}$;
5. *in a coordinate frame* $\Gamma^\mu_{\mu\nu} = \partial_\nu(\log\sqrt{|g|})$ *(this is useful in computing the Ricci curvature tensor);*
6. *in a coordinate frame* $g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = -|g|^{-1/2}\partial_\nu(|g|^{1/2}g^{\mu\nu})$ *(this is useful in computing the Ricci scalar);*

7. *in a coordinate frame* $\nabla_\mu V^\mu = |g|^{-1/2}\partial_\mu(|g|^{1/2}V^\mu)$ *(this is the covariant divergence of a contravariant vector);*

8. *in a coordinate frame* $\nabla_\sigma A_\mu{}^\sigma = |g|^{-1/2}\partial_\sigma(|g|^{1/2}A_\mu{}^\sigma) - \Gamma^\sigma_{\mu\tau}A_\sigma{}^\tau$ *(this is the covariant divergence of a rank-2 tensor);*

9. *in a coordinate frame* $\nabla_\nu A^{\mu\nu} = |g|^{-1/2}\partial_\nu(|g|^{1/2}A^{\mu\nu})$ *for every antisymmetric tensor* $A^{\mu\nu}$*;*

10. *in a coordinate frame* $\nabla_\mu\nabla^\mu\Phi = |g|^{-1/2}\partial_\mu(|g|^{1/2}g^{\mu\nu}\partial_\nu\Phi)$ *(this is the covariant D'Alambertian of a scalar).*

Solution:

Whenever possible we will establish the results in intrinsic notation.

1. This is nothing but a way to say that the metric tensor is covariantly constant, so that

$$\begin{aligned} 0 &= \nabla_\alpha g_{\mu\nu} \\ &= \partial_\alpha g_{\mu\nu} - \Gamma^\sigma_{\alpha\nu}g_{\sigma\mu} - \Gamma^\sigma_{\alpha\mu}g_{\sigma\nu} \\ &= \partial_\alpha g_{\mu\nu} - \Gamma_{\mu\alpha\nu} - \Gamma_{\nu\alpha\mu} \\ &= \partial_\alpha g_{\mu\nu} - \Gamma_{\mu\nu\alpha} - \Gamma_{\nu\mu\alpha}. \end{aligned}$$

Indeed this comes from the compatibility condition and the symmetry of the connection. We show this again by considering a basis of vectors $\{\boldsymbol{e}_\alpha\}_{\alpha=0,\dots,3}$. We have

$$\begin{aligned} \partial_\alpha(g_{\mu\nu}) &= \boldsymbol{e}_\alpha(\langle \boldsymbol{e}_\mu, \boldsymbol{e}_\nu\rangle) \\ &= \langle D(\boldsymbol{e}_\alpha, \boldsymbol{e}_\mu), \boldsymbol{e}_\nu\rangle + \langle D(\boldsymbol{e}_\alpha, \boldsymbol{e}_\nu), \boldsymbol{e}_\mu\rangle \\ &= \Gamma^\sigma_{\alpha\mu}\langle \boldsymbol{e}_\sigma, e_\nu\rangle + \Gamma^\sigma_{\alpha\nu}\langle \boldsymbol{e}_\sigma, e_\mu\rangle \\ &= \Gamma^\sigma_{\alpha\mu}g_{\sigma\nu} + \Gamma^\sigma_{\alpha\nu}g_{\sigma\mu} \\ &= \Gamma_{\nu\alpha\mu} + \Gamma_{\mu\alpha\nu} \\ &= \Gamma_{\nu\mu\alpha} + \Gamma_{\mu\nu\alpha} \\ &= \Gamma_{\mu\nu\alpha} + \Gamma_{\nu\mu\alpha}. \end{aligned}$$

2. From the definition of the inverse of the metric, $g^{\mu\nu} = (g^{-1})_{\mu\nu}$ we know that $g_{\mu\sigma}g^{\sigma\nu} = \delta^\nu_\mu$. Taking the derivative ∂_τ of both sides, we get

$$(\partial_\tau g_{\mu\sigma})g^{\sigma\nu} + g_{\mu\sigma}(\partial_\tau g^{\sigma\nu}) = 0$$

from which we get

$$g_{\mu\sigma}(\partial_\tau g^{\sigma\nu}) = -g^{\sigma\nu}(\partial_\tau g_{\mu\sigma}).$$

3. This is nothing but another way to write that the (inverse of) the metric tensor is covariantly constant, which has been proved in problem 4.2. Indeed we have that

$$0 = \nabla_\nu g^{\alpha\beta} = \nabla_\nu g^{\alpha\beta} = \partial_\nu g^{\alpha\beta} + \Gamma^\alpha_{\mu\nu}g^{\nu\beta} + \Gamma^\beta_{\mu\nu}g^{\nu\alpha}.$$

4. This useful result will be proved using the identity

$$\log(\det(\boldsymbol{A})) = \mathrm{Tr}\left(\log(\boldsymbol{A})\right),$$

which holds if $\boldsymbol{A} \in \mathrm{GL}(\mathrm{n}, \mathbb{C})$. Then, since $g = \det(g_{\mu\nu})$[2],

$$\begin{aligned}
\partial_\alpha g &= \partial_\alpha e^{\log\det(g_{\mu\nu})} \\
&= \partial_\alpha e^{\mathrm{Tr}\left(\log(g_{\mu\nu})\right)} \\
&= e^{\mathrm{Tr}\left(\log(g_{\mu\nu})\right)} \partial_\alpha \mathrm{Tr}\left(\log(g_{\mu\nu})\right) \\
&= g\mathrm{Tr}\left(\partial_\alpha \log(g_{\mu\nu})\right) \\
&= g\mathrm{Tr}\left(\sum_\beta (g^{-1})_{\mu\beta}\partial_\alpha g_{\beta\nu}\right) \\
&= g\mathrm{Tr}\left(g^{\mu\beta}\partial_\alpha g_{\beta\nu}\right) \\
&= g g^{\mu\beta}\partial_\alpha g_{\beta\mu} \\
&= g g^{\mu\nu}\partial_\alpha g_{\nu\mu} \\
&= g g^{\mu\nu}\partial_\alpha g_{\mu\nu} \\
&= -g g_{\mu\nu}\partial_\alpha g^{\mu\nu}.
\end{aligned}$$

5. From the definition of the connection coefficients in a coordinate frame we have

$$2\Gamma_{\alpha\mu\nu} = -\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu}$$

so that

$$\begin{aligned}
2\Gamma^\mu_{\mu\nu} &= g^{\alpha\mu}\Gamma_{\alpha\mu\nu} \\
&= -g^{\alpha\mu}\partial_\alpha g_{\mu\nu} + g^{\alpha\mu}\partial_\mu g_{\nu\alpha} + g^{\alpha\mu}\partial_\nu g_{\alpha\mu} \\
&= -g^{\alpha\mu}\partial_\alpha g_{\mu\nu} + g^{\mu\alpha}\partial_\alpha g_{\nu\mu} + g^{\alpha\mu}\partial_\nu g_{\alpha\mu} \\
&= -g^{\alpha\mu}\partial_\alpha g_{\mu\nu} + g^{\alpha\mu}\partial_\alpha g_{\mu\nu} + g^{\alpha\mu}\partial_\nu g_{\alpha\mu} \\
&= g^{\alpha\mu}\partial_\nu g_{\alpha\mu} \\
&= \frac{\partial_\alpha g}{g} \\
&= \frac{\partial_\alpha |g|}{|g|} \\
&= \partial_\alpha \log|g|.
\end{aligned}$$

Then by a property of the logarithm

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2}\partial_\alpha \log|g| = \partial_\alpha \log|g|^{1/2}.$$

We remember again that the above results are only valid in a coordinate frame because only in a coordinate frame we can express the connection coefficients in terms of the metric as in the starting equation.

6. From the expression for the connection coefficients in a coordinate frame we get

$$g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\alpha\beta}g^{\mu\gamma}\left(-\partial_\gamma g_{\alpha\beta} + \partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha}\right)$$

[2]Some functions on matrices (as log in this case), can be defined by power series.

$$
\begin{aligned}
&= -\frac{1}{2} g^{\mu\gamma} g^{\alpha\beta} \partial_\gamma g_{\alpha\beta} + \\
&\qquad + \frac{1}{2} g^{\mu\gamma} \left(g^{\alpha\beta} \partial_\alpha g_{\beta\gamma} + g^{\beta\alpha} \partial_\alpha g_{\gamma\beta} \right) \\
&= -\frac{1}{2} g^{\mu\gamma} \frac{\partial_\gamma g}{g} + g^{\mu\gamma} g^{\alpha\beta} \partial_\alpha g_{\beta\gamma} \\
&= -\frac{1}{2} g^{\mu\gamma} \partial_\gamma \log|g| + \partial_\alpha (g^{\mu\gamma} g^{\alpha\beta} g_{\beta\gamma}) + \\
&\qquad - (\partial_\alpha g^{\mu\gamma}) g^{\alpha\beta} g_{\beta\gamma} - g^{\mu\gamma} (\partial_\alpha g^{\alpha\beta}) g_{\beta\gamma} \\
&= -g^{\mu\gamma} \partial_\gamma (\log|g|^{1/2}) + \partial_\alpha (\delta^\mu_\beta g^{\alpha\beta}) + \\
&\qquad - (\partial_\alpha g^{\mu\gamma}) \delta^\alpha_\gamma - \delta^\mu_\beta (\partial_\alpha g^{\alpha\beta}) \\
&= -\frac{g^{\mu\nu} \partial_\nu |g|^{1/2}}{|g|^{1/2}} + \partial_\alpha g^{\alpha\mu} - \partial_\alpha g^{\mu\alpha} - \partial_\alpha g^{\alpha\mu} \\
&= -|g|^{-1/2} \left(g^{\mu\nu} \partial_\nu |g|^{1/2} + |g|^{1/2} \partial_\nu g^{\mu\nu} \right) \\
&= -|g|^{-1/2} \partial_\nu (|g|^{1/2} g^{\mu\nu}).
\end{aligned}
$$

7. We have

$$
\begin{aligned}
\nabla_\mu V^\mu &= g^{\alpha\beta} \nabla_\alpha V_\beta \\
&= g^{\alpha\beta} \partial_\alpha V_\beta - g^{\alpha\beta} \Gamma^\mu_{\alpha\beta} V_\mu \\
&= |g|^{-1/2} (|g|^{1/2} g^{\mu\nu} \partial_\mu V_\nu) + \\
&\qquad + |g|^{-1/2} \partial_\mu (|g|^{1/2} g^{\nu\mu}) V_\nu \\
&= |g|^{-1/2} (|g|^{1/2} g^{\mu\nu} \partial_\mu V_\nu + \partial_\mu (|g|^{1/2} g^{\mu\nu}) V_\nu) \\
&= |g|^{-1/2} \partial_\mu (|g|^{1/2} g^{\mu\nu} V_\nu) \\
&= |g|^{-1/2} \partial_\mu (|g|^{1/2} V^\mu).
\end{aligned}
$$

8. We start from

$$
\begin{aligned}
\nabla_\sigma A_\mu{}^\sigma &= \nabla_\sigma (A_{\mu\nu} g^{\nu\sigma}) \\
&= g^{\nu\sigma} \nabla_\sigma A_{\mu\nu} \\
&= g^{\nu\sigma} (\partial_\sigma A_{\mu\nu} - \Gamma^\alpha_{\sigma\mu} A_{\alpha\nu} - \Gamma^\alpha_{\sigma\nu} A_{\mu\alpha}) \\
&= g^{\nu\sigma} \partial_\sigma A_{\mu\nu} - g^{\nu\sigma} \Gamma^\alpha_{\sigma\mu} A_{\alpha\nu} - g^{\nu\sigma} \Gamma^\alpha_{\sigma\nu} A_{\mu\alpha} \\
&= g^{\nu\sigma} \partial_\sigma A_{\mu\nu} - \Gamma^\alpha_{\sigma\mu} A_\alpha{}^\sigma - g^{\sigma\nu} \Gamma^\alpha_{\sigma\nu} A_{\mu\alpha} \\
&= g^{\alpha\nu} \partial_\nu A_{\mu\alpha} + |g|^{-1/2} \partial_\nu (|g|^{1/2} g^{\alpha\nu}) A_{\mu\alpha} + \\
&\qquad - \Gamma^\sigma_{\mu\tau} A_\sigma{}^\tau \\
&= \frac{\left(|g|^{1/2} g^{\alpha\nu} \partial_\nu A_{\mu\alpha} + \partial_\nu (|g|^{1/2} g^{\alpha\nu}) A_{\mu\alpha} \right)}{|g|^{1/2}} + \\
&\qquad - \Gamma^\sigma_{\mu\tau} A_\sigma{}^\tau \\
&= |g|^{-1/2} \partial_\nu \left(|g|^{1/2} g^{\alpha\nu} A_{\mu\alpha} \right) - \Gamma^\sigma_{\mu\tau} A_\sigma{}^\tau \\
&= |g|^{-1/2} \partial_\sigma \left(|g|^{1/2} A_\mu{}^\sigma \right) - \Gamma^\sigma_{\mu\tau} A_\sigma{}^\tau
\end{aligned}
$$

9. For the sake of clarity we perform the covariant derivative with three distinct indices, using properly a Kronecker delta:

$$
\nabla_\nu A^{\mu\nu} = \delta^\sigma_\nu \nabla_\sigma A^{\mu\nu}
$$

$$
\begin{aligned}
&= \delta^{\sigma}_{\nu}\left(\partial_\sigma A^{\mu\nu} + \Gamma^{\mu}_{\sigma\alpha} A^{\alpha\nu} + \Gamma^{\nu}_{\sigma\alpha} A^{\mu\alpha}\right) \\
&= \partial_\nu A^{\mu\nu} + \Gamma^{\mu}_{\nu\alpha} A^{\alpha\nu} + \Gamma^{\nu}_{\nu\alpha} A^{\mu\alpha} \\
&= \partial_\nu A^{\mu\nu} + 0 + \partial_\alpha \log(|g|^{1/2}) A^{\mu\alpha} \\
&= |g|^{-1/2}|g|^{1/2}\partial_\nu A^{\mu\nu} + |g|^{-1/2}\partial_\nu |g|^{1/2} A^{\mu\nu} \\
&= |g|^{-1/2}(\partial_\nu |g|^{1/2} A^{\mu\nu}).
\end{aligned}
$$

10. We have $\nabla_\nu \Phi = \partial_\nu \Phi$ and $\nabla^\mu \Phi = g^{\mu\nu}\nabla_\nu \Phi = g^{\mu\nu}\partial_\nu \Phi$. But $\nabla^\mu \Phi$ is a contravariant vector and we can apply result 7. above, so that

$$
\begin{aligned}
\nabla_\mu \nabla^\mu \Phi &= |g|^{-1/2}\partial_\mu(|g|^{1/2}\nabla^\mu \Phi) \\
&= |g|^{-1/2}\partial_\mu(|g|^{1/2} g^{\mu\nu}\partial_\nu \Phi).
\end{aligned}
$$

We stress again that results 5., 6., 7., 8., 9. and 10. are only valid in a coordinate basis.

□

Problem 4.4 (Intrinsic and component notations) *To get some practice in passing from component to intrinsic notation, write the following expressions in intrinsic notation:*

1. $V_{\mu;\nu}V^\mu V^\nu$;
2. $V^\mu{}_{;\nu}W^\nu - W^\mu{}_{;\nu}V^\nu$;
3. $T_{\mu\nu;\alpha}U^\alpha V^\mu W^\nu$;
4. $U^{\mu;\alpha}V_{\alpha;\sigma}W^\sigma$;

Solution:

1. The result is a scalar (no free index); in particular

$$
\begin{aligned}
V_{\mu;\nu}V^\mu V^\nu &= g_{\rho\mu}\nabla_\nu V^\rho V^\mu V^\nu \\
&= g_{\rho\mu}(\partial_\nu V^\rho + \Gamma^{\rho}_{\nu\alpha}V^\alpha)V^\mu V^\nu \\
&= \langle \boldsymbol{e}_\rho, \boldsymbol{e}_\mu \rangle\, \boldsymbol{e}_\nu(V^\rho)V^\mu V^\nu + \\
&\qquad + \langle D(\boldsymbol{e}_\nu, \boldsymbol{e}_\alpha), \boldsymbol{e}_\mu \rangle\, V^\alpha V^\mu V^\nu \\
&= \langle \boldsymbol{e}_\rho, V^\mu \boldsymbol{e}_\mu \rangle\, V^\nu \boldsymbol{e}_\nu(V^\rho) + \\
&\qquad + \langle D(V^\nu \boldsymbol{e}_\nu, \boldsymbol{e}_\alpha), V^\mu \boldsymbol{e}_\mu \rangle\, V^\alpha \\
&= \langle \boldsymbol{e}_\alpha, \boldsymbol{V} \rangle\, \boldsymbol{V}(V^\alpha) + \langle V^\alpha D(\boldsymbol{V}, \boldsymbol{e}_\alpha), \boldsymbol{V} \rangle \\
&= \langle \boldsymbol{V}(V^\alpha)\boldsymbol{e}_\alpha + V^\alpha D(\boldsymbol{V}, \boldsymbol{e}_\alpha), \boldsymbol{V} \rangle \\
&= \langle D(\boldsymbol{V}, \boldsymbol{V}), \boldsymbol{V} \rangle
\end{aligned}
$$

2. The expression is a contravariant vector (one free upper index, μ), which we write as

$$\boldsymbol{A} = (V^\mu{}_{;\nu}W^\nu - W^\mu{}_{;\nu}V^\nu)\boldsymbol{e}_\mu.$$

Let us consider

$$V^\mu{}_{;\nu}W^\nu \boldsymbol{e}_\mu :$$

we have

$$\begin{aligned}
V^{\mu}{}_{;\nu}W^{\nu}\boldsymbol{e}_{\mu} &= \partial_{\nu}V^{\mu}W^{\nu}\boldsymbol{e}_{\mu} + \Gamma^{\mu}_{\nu\alpha}V^{\alpha}W^{\nu}\boldsymbol{e}_{\mu} \\
&= W^{\nu}\boldsymbol{e}_{\nu}(V^{\mu})\boldsymbol{e}_{\mu} + \boldsymbol{E}^{\mu}(D(\boldsymbol{e}_{\nu},\boldsymbol{e}_{\alpha}))V^{\alpha}W^{\nu}\boldsymbol{e}_{\mu} \\
&= \boldsymbol{W}(V^{\mu})\boldsymbol{e}_{\mu} + \boldsymbol{E}^{\mu}(D(W^{\nu}\boldsymbol{e}_{\nu},\boldsymbol{e}_{\alpha}))V^{\alpha}\boldsymbol{e}_{\mu} \\
&= \boldsymbol{W}(V^{\mu})\boldsymbol{e}_{\mu} + D(\boldsymbol{W},\boldsymbol{e}_{\mu})V^{\mu} \\
&= D(\boldsymbol{W},V^{\mu}\boldsymbol{e}_{\mu}) \\
&= D(\boldsymbol{W},\boldsymbol{V}).
\end{aligned}$$

To get the second term we exchange $\boldsymbol{V}$ and $\boldsymbol{W}$. The two results altogether thus give

$$\boldsymbol{A} = D(\boldsymbol{W},\boldsymbol{V}) - D(\boldsymbol{V},\boldsymbol{W}) = [\boldsymbol{W},\boldsymbol{V}] = \pounds_{W}V.$$

3. The expression is a scalar (no free indices) and we have

$$\begin{aligned}
T_{\mu\nu;\alpha}U^{\alpha}V^{\mu}W^{\nu} &= D(\boldsymbol{e}_{\alpha},\boldsymbol{T})_{\mu\nu}U^{\alpha}V^{\mu}W^{\nu} \\
&= D(U^{\alpha}\boldsymbol{e}_{\alpha},\boldsymbol{T})(\boldsymbol{V},\boldsymbol{W}) \\
&= D(\boldsymbol{U},\boldsymbol{T})(\boldsymbol{V},\boldsymbol{W}).
\end{aligned}$$

4. The expression is a contravariant vector (one free upper index). We can rewrite it as follows:

$$\begin{aligned}
U^{\mu;\alpha}V_{\alpha;\sigma}W^{\sigma} &= g^{\alpha\rho}U^{\mu}{}_{;\rho}V_{\alpha;\sigma}W^{\sigma} \\
&= U^{\mu}{}_{;\rho}V^{\rho}{}_{;\sigma}W^{\sigma} \\
&= U^{\mu}{}_{;\rho}Z^{\rho},
\end{aligned}$$

where we define $Z^{\rho} = V^{\rho}{}_{;\sigma}W^{\sigma}$. We can use the result in 2. for each of these expressions. In particular

$$\boldsymbol{Z} = Z^{\rho}\boldsymbol{e}_{\rho} = D(\boldsymbol{W},\boldsymbol{V})$$

and thus

$$\begin{aligned}
U^{\mu;\alpha}V_{\alpha;\sigma}W^{\sigma}\boldsymbol{e}_{\mu} &= U^{\mu}{}_{\rho}Z^{\rho}\boldsymbol{e}_{\mu} \\
&= D(\boldsymbol{Z},\boldsymbol{U}) \\
&= D(D(\boldsymbol{W},\boldsymbol{V}),\boldsymbol{U}).
\end{aligned}$$

This completes the proofs.

□