Chapter 3

Differential Geometry

3.1 Differentiable Manifold

Definition 3.1 (Locally Euclidean Hausdorff space)
Let $\mathscr{M}$ be a topological Hausdorff Space. $\mathscr{M}$ is locally Euclidean *of dimension m if $\forall \mathtt{m} \in \mathscr{M}$ there exists a neighborhood $U \subset \mathscr{M}$ and an homeomorphism $\phi : U \longrightarrow A \subset \mathbb{R}^m$.*

ϕ is called a *coordinate application* and the couple (U, ϕ) a *coordinate system* on $\mathscr{M}$ in the neighborhood of $\mathtt{m}$. If we take the i-th canonical projection on $\mathbb{R}^m$, r_i, $x_i = r_i \circ \phi$ is the i-th *coordinate function* on $\mathscr{M}$ in the neighborhood of $\mathtt{m}$.

Definition 3.2 (Differentiable Structure)
A C^∞ differentiable structure *on a locally Euclidean, Hausdorff topological space $\mathscr{M}$ of dimension m is a collection of coordinate systems $\mathcal{F} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathcal{A}}$ which satisfies the following properties:*

1. *the $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ are a cover of $\mathscr{M}$, i.e. $\bigcup_{\alpha \in \mathcal{A}} U_\alpha = \mathscr{M}$;*

2. *for every choice of ϕ_α and ϕ_β, $\phi_\alpha \circ \phi_\beta^{-1}$ is $C^\infty(\mathbb{R}^m, \mathbb{R}^m)$, i.e. the structure is* compatible.

A differentiable structure $\mathcal{F}$ which is *maximal* is called an *atlas*.

Definition 3.3 (Differentiable Manifold)
A differentiable manifold *of dimension d is a couple $(\mathscr{M}, \mathcal{F})$ with:*

$\mathscr{M}$ a topological, locally Euclidean Hausdorff space which is second countable;

$\mathcal{F}$ an atlas for $\mathscr{M}$.

If omitting the specification of the differentiable structure will not compromit clarity, we are going to understand it implicitly, i.e. we will use equivalently sentences like "*Let $\mathscr{M}$ be a manifold*" and "*Let $(\mathscr{M}, \mathcal{F})$ be a manifold*".

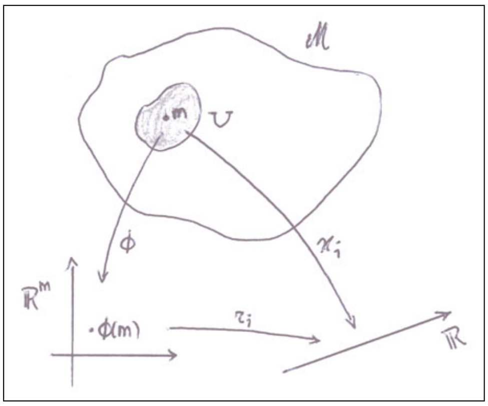

Figure 3.1: Coordinate maps and function on a locally euclidean Hausdorff space.

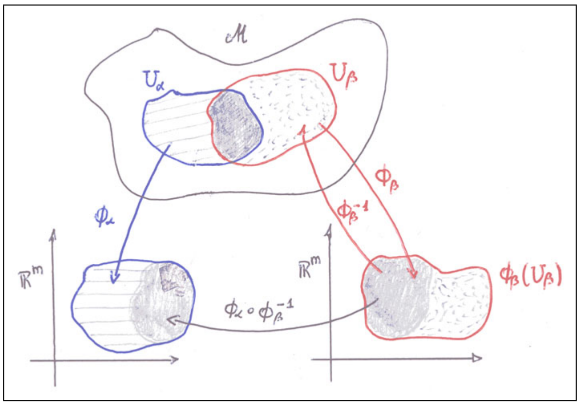

Figure 3.2: Compatibility condition for a differentiable structure.

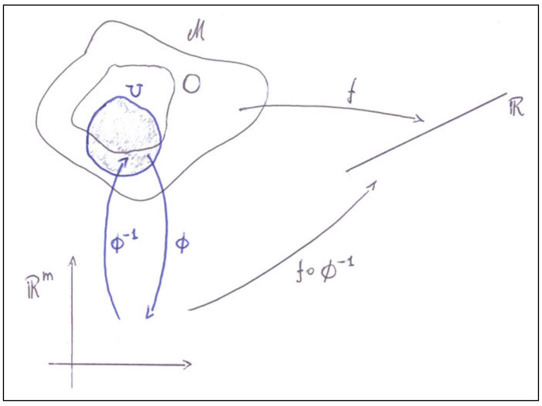

Figure 3.3: Differentiable function on a manifold.

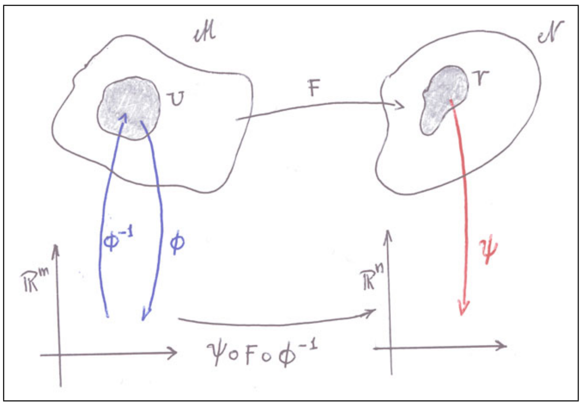

Figure 3.4: Differentiable map between manifolds.

3.2 Maps on Manifolds

Definition 3.4 (C^∞ function on a manifold)
Let $(\mathscr{M}, \mathcal{F})$ be a differentiable manifold of dimension m and O an open subset of $\mathscr{M}$. A function $f : O \longrightarrow \mathbb{R}$ is differentiable of class C^∞ *if* $\forall (U, \phi) \in \mathcal{F}$ *then* $f \circ \phi^{-1}$ *is* $C^\infty(\mathbb{R}^m, \mathbb{R})$.

Definition 3.5 (C^∞ map between manifolds)
Let us consider $(\mathscr{M}, \mathcal{F})$ and $(\mathscr{N}, \mathcal{G})$, two differentiable manifolds of dimension m and n respectively. A map $F : \mathscr{M} \longrightarrow \mathscr{N}$ is differentiable of class C^∞ *if for every choice of $(U, \phi) \in \mathcal{F}$ and $(V, \psi) \in \mathcal{G}$ $\psi \circ f \circ \phi^{-1}$ is $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$.*

Definition 3.6 (Smooth curve on a manifold)
Let us consider a manifold $(\mathscr{M}, \mathcal{F})$. A smooth curve on $\mathscr{M}$ is a differentiable map

$$\sigma : [a, b] \longrightarrow \mathscr{M}$$

such that $\sigma(t) \in \mathscr{M}$. The tangent vector to the curve is denoted by $\dot{\boldsymbol{\sigma}}(t)$, which is defined as

$$\dot{\boldsymbol{\sigma}}(t) = d\sigma\rceil_t \left(\frac{d}{dr}\right\rceil_t \Big).$$

Remember that the differential of $\sigma(t)$ is a map

$$d\sigma\rceil_t : \mathbb{R}_t \cong \mathbb{R} \longrightarrow \mathscr{M}_{\sigma(t)},$$

which maps tangent vectors in $\mathbb{R}_t$ into tangent vectors of $\mathscr{M}_{\sigma(t)}$.

3.3 Partition of unity

Definition 3.7 (Differentiable partition of unity)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold. A differentiable partition of unity *is a couple $(\mathscr{R}, \mathcal{P})$ where:*

1. *$\mathscr{R}$ is a locally finite open cover of $\mathcal{M}$;*
2. *$\mathcal{P}$ is a collection of functions*

$$\mathcal{P} = \{f_V : \mathcal{M} \longrightarrow \mathbb{R} | V \in \mathscr{R},\ f \quad \text{differentiable}\}$$

   (a) $f_V \geq 0$, $\forall V \in \mathscr{R}$;
   (b) $\operatorname{supp}(f_V) \subset V$;
   (c) $\sum_{V \in \mathscr{R}} f_V = 1$.

We see that the sum is finite because $\mathscr{R}$ is a locally finite open cover of $\mathscr{M}$. Thus $\forall \mathtt{m} \in \mathscr{M}$ it is possible to find a neighborhood P which intersects only a finite number of $V \in \mathscr{R}$. In that neighborhood the sum is thus restricted only to these V's.

Proposition 3.1 (Existence of partition of unity)
Let $(\mathscr{M}, \mathcal{F})$ be a paracompact differentiable manifold and let $\mathscr{U}$ be an open cover of $\mathscr{M}$. There exists a partition of unity $(\mathscr{R}, \mathcal{P})$ where $\mathscr{R}$ is a locally finite open refinement of $\mathscr{U}$.

We will say that the partition of unity $(\mathscr{R}, \mathcal{P})$ is subordinated to the cover $\mathscr{U}$. The paracompactness is required to obtain the open locally finite refinement $\mathscr{R}$ starting from $\mathscr{U}$.

3.4 Tangent Space

Definition 3.8 ($C^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R})$)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold and let $\mathtt{m} \in \mathscr{M}$. We define $C^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R})$ as the set of all functions defined on a neighborhood of $\mathtt{m}$ and with real values.

Definition 3.9 (Germs of functions around $\mathtt{m} \in \mathscr{M}$)
Let us consider $f, g \in C^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R})$. We define an equivalence relation, $\sim$, in $C^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R})$ as follows: $[f \sim g] \Leftrightarrow [f = g$ in a neighborhood of $\mathtt{m}]$. A germ *of functions around $\mathtt{m}$ is an element, $\hat{f} \stackrel{\text{def.}}{=} [f]$, of the $\mathbb{R}$-algebra $\tilde{C}^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R}) = C^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R}) / \sim$.*

Definition 3.10 (Tangent vector at $\mathtt{m} \in \mathscr{M}$)
A tangent vector $\boldsymbol{v}$ *at* $m \in \mathscr{M}$ *is a linear map* $\boldsymbol{v} : \tilde{C}^\infty(\mathscr{M}, m, \mathbb{R}) \longrightarrow \mathbb{R}$ *such that*

$$\forall \hat{f}, \hat{g} \in \tilde{C}^\infty(\mathscr{M}, m, \mathbb{R}) \boldsymbol{v}(\hat{f}\hat{g}) = \boldsymbol{v}(\hat{f})\hat{g}(m) + \boldsymbol{v}(\hat{g})\hat{f}(m).$$

$\boldsymbol{v}$ is a derivation of $\tilde{C}^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R})$.

Definition 3.11 (Tangent space at $\mathtt{m} \in \mathscr{M}$)
The tangent space *at* $m \in \mathscr{M}$ *is*

$$M_m = T_m\mathscr{M} \stackrel{\text{def.}}{=} \{\boldsymbol{v} | \boldsymbol{v} \text{is a tangent vector at} m \in \mathscr{M}\}.$$

The name "space" is justified by the following

Proposition 3.2 (Dimension and coordinate basis of $\mathscr{M}_\mathtt{m}$)
$\mathscr{M}_m$ *is a vector space of dimension* $\dim(\mathscr{M}_m) = \dim(\mathscr{M})$ *and*

$$\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1,\ldots,m}$$

is a basis of $\mathscr{M}_m$, *with*

$$\left.\frac{\partial}{\partial x_i}\right]_m (f) \stackrel{\text{def.}}{=} \left.\frac{\partial}{\partial r_i}\right]_{\phi(m)} (f \circ \phi^{-1}).$$

The basis

$$\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1,\ldots,m}$$

is called the *coordinate basis* of $\mathscr{M}_\mathtt{m}$.

Proof:

The proof that $\mathscr{M}_\mathtt{m}$ is a vector space is left to the reader: all the axioms of a vector space structure are satisfied when the vector space operations are defined as follows:

$$\begin{aligned}(\boldsymbol{v}+\boldsymbol{w})(\hat{f}) &\stackrel{\text{def.}}{=} \boldsymbol{v}(\hat{f}) + \boldsymbol{w}(\hat{f}) \quad , \quad \forall \boldsymbol{v}, \boldsymbol{w} \in (\mathscr{M}_\mathtt{m}), \quad \hat{f} \in \tilde{C}^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R}) \\ (\lambda \cdot \boldsymbol{v})(\hat{f}) &\stackrel{\text{def.}}{=} \lambda \boldsymbol{v}(\hat{f}) \quad , \quad \forall \boldsymbol{v} \in (\mathscr{M}_\mathtt{m}), \quad \lambda \in \mathbb{R}, \quad \hat{f} \in \tilde{C}^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R}),\end{aligned}$$

of course the operations on the left-hand sides of the above definitions are the operations that define the vector space structure of $\mathscr{M}_\mathtt{m}$, whereas on the right hand sides the sum and product are the ordinary sum and product of real numbers.

We will now prove that $\{\partial/\partial x_i]_\mathtt{m}\}_{i=1,\ldots,m}$ is actually a basis of $\mathscr{M}_\mathtt{m}$, which will also give a proof of the statement about its dimension. Let us thus consider a coordinate map $\phi : \mathscr{M} \longrightarrow \mathbb{R}^m$ for $\mathscr{M}$ around $\mathtt{m}$ with $\phi(\mathtt{m}) = (a_1, \ldots, a_m) \in \mathbb{R}^m$. For $f \in C^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R})$ we define the map $F = f \circ \phi^{-1}$, for which the following chain of equalities holds:

$$\begin{aligned} F(\vec{r}) - F(\vec{a}) &= \int_0^1 \frac{d}{ds} F(s(\vec{r}-\vec{a}) + \vec{a}) ds \\ &= \sum_i^{1,m} \int_0^1 \frac{\partial}{\partial r_i} F(s(\vec{r}-\vec{a}) + \vec{a})(r_i - a_i) ds \\ &= \sum_i^{1,m} (r_i - a_i) \int_0^1 \frac{\partial}{\partial r_i} F(s(\vec{r}-\vec{a}) + \vec{a}) ds \\ &= \sum_i^{1,m} (r_i - a_i) H_i(\vec{r}), \end{aligned} \tag{3.1}$$

where the functions H_i are C^∞ because so is F.
Composing with ϕ the above relation we obtain that $\forall f \in C^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R})$ we can find functions $h_1, \ldots, h_m \in C^\infty(\mathscr{M}, \mathtt{m}, \mathbb{R})$ such that

$$f(x) = f(\mathtt{m}) + \sum_i^{1,m} (x_i - a_i) h_i(x).$$

Of course this result implies that

$$\left.\frac{\partial}{\partial x_i}\right\rceil_{\mathtt{m}} (f) = h_i(\mathtt{m}).$$

Let us then consider a tangent vector $\boldsymbol{v} \in \mathscr{M}_{\mathtt{m}}$. We have

$$\begin{aligned} \boldsymbol{v}(f) &= \boldsymbol{v}(f(\mathtt{m}) + \sum_i^{1,m} (x_i - a_i) h_i) \\ &= \sum_i^{1,m} \boldsymbol{v}(x_i) h_i(\mathtt{m}) \\ &= \left(\sum_i^{1,m} \boldsymbol{v}(x_i) \left.\frac{\partial}{\partial x_i}\right\rceil_{\mathtt{m}} \right)(f). \end{aligned} \tag{3.2}$$

Thus the $\{\partial/\partial x_i\rceil_{\mathtt{m}}\}_{i=1,\ldots,m}$ are a system of generators for $\mathscr{M}_{\mathtt{m}}$.
Let us consider a vanishing linear combination,

$$0 = \sum_j^{1,m} \lambda_j \left.\frac{\partial}{\partial x_j}\right\rceil_{\mathtt{m}};$$

then

$$0 = \left(\sum_j^{1,m} \lambda_j \left.\frac{\partial}{\partial x_j}\right\rceil_{\mathtt{m}} \right)(x_i) = \lambda_i \quad , \quad \forall i = 1, \ldots, m,$$

so that the system of generators is actually linearly independent. It is thus a basis.

□

3.5 Cotangent Space and the differential

Definition 3.12 (Cotangent vector at $\mathtt{m} \in \mathscr{M}$)
A cotangent vector $\boldsymbol{v}$ *at* $\mathtt{m} \in \mathscr{M}$ *is a linear map* $\boldsymbol{\omega} : \mathscr{M}_{\mathtt{m}} \longrightarrow \mathbb{R}$.

Definition 3.13 (Cotangent space at $\mathtt{m} \in \mathscr{M}$)
The cotangent space *at* $\mathtt{m} \in \mathscr{M}$ *is the vector space,* $\mathscr{M}^*_{\mathtt{m}}$*, dual to* $\mathscr{M}_{\mathtt{m}}$*, i.e. the vector space of all linear maps from* $\mathscr{M}_{\mathtt{m}}$ *into* $\mathbb{R}$.

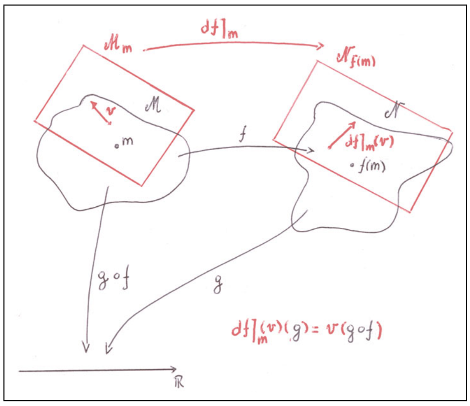

Figure 3.5: Differential of a map between manifolds.

Definition 3.14 (Differential of a function between manifolds)
Let $f : \mathscr{M} \longrightarrow \mathscr{N}$ be a differentiable map between two differentiable manifolds. Let us consider $\mathtt{m} \in \mathscr{M}$, so that $f(\mathtt{m}) \in \mathscr{N}$, and a map

$$df\rceil_{\mathtt{m}} : \mathscr{M}_{\mathtt{m}} \longrightarrow \mathscr{N}_{f(\mathtt{m})}$$

such that

$$\forall \boldsymbol{v} \in \mathscr{M} \forall \hat{g} \in \tilde{C}^{\infty}(\mathscr{M}, \mathtt{m}, \mathbb{R}) df(\boldsymbol{v})(\hat{g}) \stackrel{\text{def.}}{=} \boldsymbol{v}(g \circ f).$$

$df\rceil_{\mathtt{m}}$ *is the* differential *of f in $\mathtt{m}$.*

Proposition 3.3 (Coordinate representation of $df\rceil_{\mathtt{m}}$)
Let $f : \mathscr{M} \longrightarrow \mathscr{N}$ be a differentiable map between two differentiable manifolds. Given (U, ϕ) chart around $\mathtt{m} \in \mathscr{M}$ and (V, ψ) chart around $f(\mathtt{m}) \in \mathscr{N}$ the differential of f is represented by the Jacobean Matrix

$$\left(\frac{\partial f_j}{\partial x_i}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}} .$$

Proof:

Let us call $\{x_i\}_{i=1,\dots,m}$ and $\{y_j\}_{j=1,\dots,n}$ the coordinate functions associated to the coordinate maps (U, ϕ) and (V, ψ) around $\mathtt{m} \in \mathscr{M}$ and $f(\mathtt{m}) \in \mathscr{N}$ respectively. If $g \in C^{\infty}(\mathscr{N}, f(\mathtt{m}), \mathbb{R})$ then $g \circ f \in C^{\infty}(\mathscr{M}, \mathtt{m}, \mathbb{R})$ and by definition of $df\rceil_{\mathtt{m}}$:

$$\begin{aligned}
df\left(\frac{\partial}{\partial x_i}\bigg\rceil_{\mathtt{m}}\right)(g) &= \frac{\partial}{\partial x_i}\bigg\rceil_{\mathtt{m}}(g \circ f) \\
&= \frac{\partial}{\partial r_i}\bigg\rceil_{\phi(\mathtt{m})}(g \circ f \circ \phi^{-1}) \\
&= \frac{\partial}{\partial r_i}\bigg\rceil_{\phi(\mathtt{m})}(g \circ \psi^{-1} \circ \psi \circ f \circ \phi^{-1}) \\
&= \sum_{j}^{1,n} \frac{\partial}{\partial s_j}\bigg\rceil_{\psi(f(\mathtt{m}))}(g \circ \psi^{-1}) \frac{\partial}{\partial r_i}\bigg\rceil_{\phi(\mathtt{m})}(s_j \circ \psi \circ f \circ \phi^{-1})
\end{aligned}$$

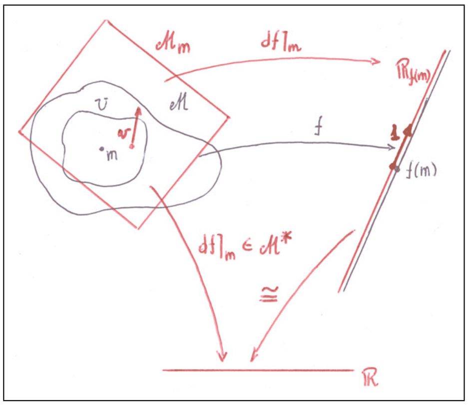

Figure 3.6: Differential of a function.

$$= \sum_j^{1,n} \frac{\partial}{\partial s_j}\bigg]_{\psi(f(\mathtt{m}))} (g \circ \psi^{-1}) \frac{\partial}{\partial r_i}\bigg]_{\phi(\mathtt{m})} (y_j \circ f \circ \phi^{-1})$$

$$= \sum_j^{1,n} \frac{\partial}{\partial x_i}\bigg]_{\mathtt{m}} (y_j \circ f) \frac{\partial}{\partial y_j}\bigg]_{f(\mathtt{m})} (g)$$

$$= \left(\sum_j^{1,n} \frac{\partial}{\partial x_i}\bigg]_{\mathtt{m}} (y_j \circ f) \frac{\partial}{\partial y_j}\bigg]_{f(\mathtt{m})} \right) (g).$$

Thus

$$df\left(\frac{\partial}{\partial x_i}\bigg]_{\mathtt{m}} \right) = \sum_j^{1,n} \frac{\partial}{\partial x_i}\bigg]_{\mathtt{m}} (y_j \circ f) \frac{\partial}{\partial y_j}\bigg]_{f(\mathtt{m})}$$

or

$$(df)_{i,j} = \left(\frac{\partial}{\partial x_i}\bigg]_{\mathtt{m}} (y_j \circ f) \right)_{i,j} \stackrel{\text{def.}}{=} \text{Jacobean matrix of } f.$$

□

We consider now the special case of a function $f : U \subset \mathcal{M} \longrightarrow \mathbb{R}$. Then, given $\mathtt{m} \in U$, the differential of f is a linear map $df]_{\mathtt{m}} : \mathscr{M}_{\mathtt{m}} \longrightarrow \mathbb{R}_{f(\mathtt{m})}$. The tangent space $\mathbb{R}_{f(\mathtt{m})}$ with basis $\partial/(\partial\mathbf{1})]_{f(\mathtt{m})}$, where $\mathbf{1}$ is the identity map (natural projection) on $\mathbb{R}$. Of course $\mathbb{R}_{f(\mathtt{m})} \cong \mathbb{R}$ and we use the identification $\partial/(\partial\mathbf{1})]_{f(\mathtt{m})} \equiv 1$. In this way we have that $df]_{\mathtt{m}} \in \mathscr{M}^*_{\mathtt{m}}$ is an element of the cotangent space at $\mathtt{m} \in \mathscr{M}$.

Proposition 3.4 (Coordinate basis in $\mathscr{M}^*_{\mathtt{m}}$)
Given a chart (U, ϕ), the differentials of the coordinate functions on $U \subset \mathscr{M}$ are a basis of the cotangent space $\mathscr{M}^*_{\mathtt{m}}$ in $\mathtt{m} \in U$ at $\mathscr{M}$.

Proof:

Consider $\{dx_i]_{\mathtt{m}}, i = 1, \ldots, m\}$. Since $x_i : U \subset \mathscr{M} \longrightarrow \mathbb{R}$, where we remember that $x_i = r_i \circ \phi$, we can use the observation of the special case above. This means that $dx_i\rceil_{\mathtt{m}} \in \mathscr{M}^*_{\mathtt{m}}$. Moreover for $i = 1, \dots, m$, $i = 1, \dots, m$, we have

$$dx_i\rceil_{\mathtt{m}}\left(\frac{\partial}{\partial x_j}\bigg\rceil_{\mathtt{m}}\right) = \frac{\partial}{\partial x_j}\bigg\rceil_{\mathtt{m}} \hat{x}_i = \delta_{ij}$$

so that $\{dx_i\rceil_{\mathtt{m}},\ i = 1, \dots, m\}$ is the basis dual to the basis $\{\partial/(\partial x_i)\rceil_{\mathtt{m}},\ i = 1, \dots, m\}$ of $\mathscr{M}_{\mathtt{m}}$.

□

$\{dx_i\rceil_{\mathtt{m}}, i = 1, \dots, m\}$ is the *coordinate basis* in $\mathscr{M}_{\mathtt{m}}$ and the $dx_i\rceil_{\mathtt{m}}$ are called 1-forms in $\mathtt{m} \in \mathscr{M}$.

Proposition 3.5 (Coordinate expression of the differential)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold and $f : U \subset \mathscr{M} \longrightarrow \mathbb{R}$ a function on $\mathscr{M}$. Let $\mathtt{m} \in U$, (U, ϕ) a chart for $\mathscr{M}$. We have

$$df\rceil_{\mathtt{m}} = \sum_i^{1,m} \frac{\partial f}{\partial x_i}\bigg\rceil_{\mathtt{m}} dx_i\rceil_{\mathtt{m}}.$$

Proof:

This result can be proved applying both sides to $\partial/(\partial x_j)$:

$$\begin{aligned}
&\frac{\partial f}{\partial x_j}\bigg\rceil_{\mathtt{m}} = \\
&\frac{\partial}{\partial x_j}\bigg\rceil_{\mathtt{m}}(f) = \quad df\rceil_{\mathtt{m}}\ \frac{\partial}{\partial x_j}\bigg\rceil_{\mathtt{m}} &&= \sum_i^{1,m} \frac{\partial f}{\partial x_i}\bigg\rceil_{\mathtt{m}} dx_i\rceil_{\mathtt{m}}\left(\frac{\partial}{\partial x_j}\bigg\rceil_{\mathtt{m}}\right) \\
& &&= \sum_i^{1,m} \frac{\partial f}{\partial x_i}\bigg\rceil_{\mathtt{m}} \delta_{ij} \\
& &&= \frac{\partial f}{\partial x_j}\bigg\rceil_{\mathtt{m}}
\end{aligned} \tag{3.3}$$

and observing that they yield the same result. By linearity the right hand side thus acts in the same way as the left hand side on every tangent vector and the equality follows.

□

3.6 Forms and Tensor at a point

Tangent and cotangent vectors are not the only concepts that can be defined at a given point $\mathtt{m}$ of a manifold $\mathscr{M}$. In particular, we have seen that the tangent space at $\mathtt{m} \in \mathscr{M}$ is a vector space: thus all structure that are defined on a vector space V are defined at a given point of a manifold by the identification $V = \mathscr{M}_{\mathtt{m}}$.

Definition 3.15 (Form at $\mathtt{m} \in \mathscr{M}$)
A k-form $\boldsymbol{\omega}$ at $\mathtt{m} \in \mathscr{M}$ is an element $\boldsymbol{\omega} \in \Lambda^k(\mathscr{M}_{\mathtt{m}})$.

Thus it is a k-linear alternating map $\omega : (\mathscr{M}_{\mathtt{m}})^k \longrightarrow \mathbb{R}$, acting on a k-ple, ($\boldsymbol{v}_1$, ..., $\boldsymbol{v}_k$), of vectors tangent to $\mathscr{M}$ at $\mathtt{m}$. An analogous definition holds for tensors.

Definition 3.16 (Tensor at $\mathtt{m} \in \mathscr{M}$)
An (r, s)-tensor $\boldsymbol{T}$ at $\mathtt{m} \in \mathscr{M}$ is an element $\boldsymbol{T} \in T^r_s(\mathscr{M}_{\mathtt{m}})$.

$\boldsymbol{T}$ is a linear map that associates to r tangent vectors and s one-forms at $\mathtt{m} \in \mathscr{M}$ a real number.

All properties of forms and tensors over a vector space can be generalized for forms and tensors at a given point of a differentiable manifold. Basis for forms and tensors can be obtained from fixed basis in $\mathscr{M}_{\mathtt{m}}$ and in particular for coordinate basis. Symmetrization, anti-symmetrization and all operators defined on tensors clearly also holds. The only important remark is that in general, there is no preferred way to relate forms and tensors at different points of a manifold (the same is true of vectors, of course).

3.7 Bundles

Definition 3.17 (Vector bundle)
Let $\mathscr{M}$ and $\mathscr{B}$ be two manifolds, V a vector space and $\pi : \mathscr{B} \longrightarrow \mathscr{M}$ a differential map such that:

1. *π is surjective;*
2. *$\forall \mathtt{m} \in \mathscr{M}$ there exists $U \subset \mathscr{M}$ neighborhood of $\mathtt{m}$ such that $\pi^{-1}(U)$ is isomorphic with $U \times V$.*

Then $\mathscr{B}$ is called a vector bundle *over $\mathscr{M}$.*

$\mathscr{M}$ is called the *base space*, V is the *fiber*

Definition 3.18 (Section of a vector bundle)
Let $\mathscr{B}$ be a vector bundle over $\mathscr{M}$. A map $\Sigma : \mathscr{M} \longrightarrow \mathscr{B}$ such that

$$\Sigma \circ \pi = \mathbb{I}_{\mathscr{M}}$$

is called a section *of $\mathscr{B}$.*

Definition 3.19 (Tangent bundle)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold and

$$T(\mathscr{M}) \stackrel{\text{def.}}{=} \bigcup_{\mathtt{m} \in \mathscr{M}} \mathscr{M}_{\mathtt{m}}.$$

$T(\mathscr{M})$ together with the canonical projection

$$\pi : T(\mathscr{M}) \longrightarrow \mathscr{M}$$

is the tangent bundle *over $\mathscr{M}$.*

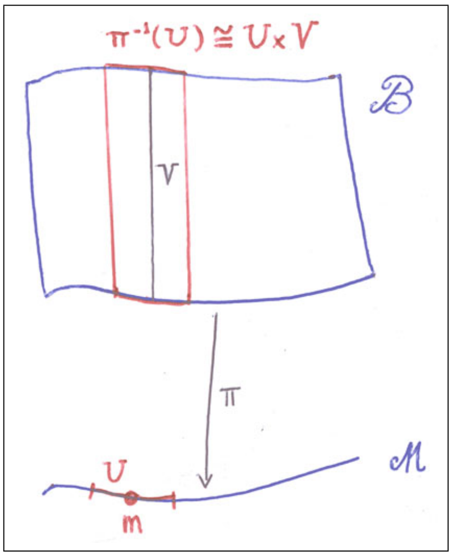

Figure 3.7: Vector bundle.

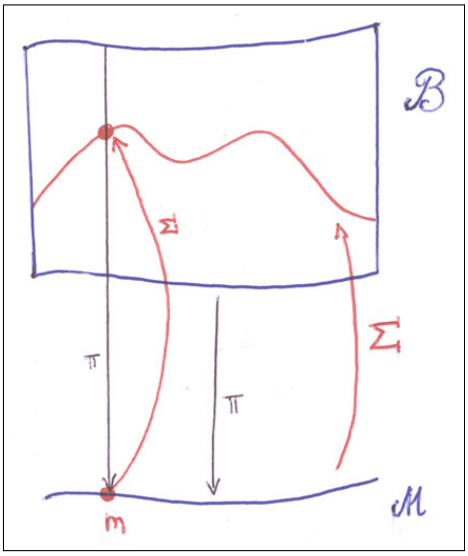

Figure 3.8: Section of a vector bundle.

Definition 3.20 (Cotangent bundle)
Let $(\mathscr{M}, \mathcal{F})$ *be a manifold and*

$$T^*(\mathscr{M}) \stackrel{\text{def.}}{=} \bigcup_{m \in \mathscr{M}} \mathscr{M}^*_m.$$

$T^*(\mathscr{M})$ *together with the canonical projection*

$$\pi^* : T^*(\mathscr{M}) \longrightarrow \mathscr{M}$$

is the cotangent bundle *over* $\mathscr{M}$.

Definition 3.21 (Tensor bundle of the (r, s) type)
Let $(\mathscr{M}, \mathcal{F})$ *be a manifold and*

$$T^r_s(\mathscr{M}) \stackrel{\text{def.}}{=} \bigcup_{m \in \mathscr{M}} T^r_s(\mathscr{M}_m).$$

$T^r_s(\mathscr{M})$ *together with the canonical projection*

$$\pi^r_s : T^r_s(\mathscr{M}) \longrightarrow \mathscr{M}$$

is the (r, s) tensor bundle *over* $\mathscr{M}$.

Definition 3.22 (Exterior k-bundle)
Let $(\mathscr{M}, \mathcal{F})$ *be a manifold and*

$$\Lambda^k(\mathscr{M}) = \bigcup_{m \in \mathscr{M}} \Lambda^k(\mathscr{M}_m).$$

$\Lambda^k(\mathscr{M})$ *together with the canonical projection*

$$\pi_{\Lambda^k} : \Lambda^k(\mathscr{M}) \longrightarrow \mathscr{M}$$

is the exterior k-bundle *over* $\mathscr{M}$.

Definition 3.23 (Exterior bundle)
Let $(\mathscr{M}, \mathcal{F})$ *be a manifold and*

$$\Lambda(\mathscr{M}) = \bigcup_{m \in \mathscr{M}} \Lambda(\mathscr{M}_m).$$

$\Lambda(\mathscr{M})$ *together with the canonical projection*

$$\pi_{\Lambda} : \Lambda(\mathscr{M}) \longrightarrow \mathscr{M}$$

is the exterior k-bundle *over* $\mathscr{M}$.

Proposition 3.6 (Vector bundles as differentiable manifolds)
$T(\mathscr{M})$, $T^*(\mathscr{M})$, $T^r_s(\mathscr{M})$, $\Lambda^k(\mathscr{M})$, $\Lambda(\mathscr{M})$ *are differentiable manifolds of dimension* $2m$, $2m$, m^{r+s+1}, $m + \binom{m}{k}$, $m + 2^m$ *respectively.*

Proof:

We will sketch the proof for the case of $T(\mathscr{M})$. In particular $T(\mathscr{M})$ is a topological space by choosing

$$\{\pi^{-1}(U) | (U,\phi) \text{coordinate system for} \mathscr{M}\}$$

as a basis for a topology of $T(\mathscr{M})$ (thus $\pi^{-1}(U)$ is open by definition). Moreover for all (U,ϕ) coordinate systems of $\mathscr{M}$ we define a coordinate system on $T(\mathscr{M})$, called $(\tilde{U},\tilde{\phi})$, as follows:

$\tilde{U}$ is the inverse image of U under the natural projection π: $\tilde{U} = \pi^{-1}(U)$;

$\tilde{\phi}$, which acts on a vector at a point, is a map

$$\tilde{\phi} : \pi^{-1}(U) \longrightarrow \mathbb{R}^{2m}$$

such that, if $\boldsymbol{v} \in \mathscr{M}_{\mathtt{m}}$, $\tilde{\phi}(\boldsymbol{v}) \stackrel{\text{def.}}{=} (x_1(\pi(\boldsymbol{v})), \ldots, x_m(\pi(\boldsymbol{v})), dx_1\rceil_{\mathtt{m}}(\boldsymbol{v}), \ldots, dx_m\rceil_{\mathtt{m}}(\boldsymbol{v}))$: the first m-components of the coordinates are the coordinates of $\mathtt{m} = \pi(\boldsymbol{v})$, whereas the second m-components are the coordinates of $\boldsymbol{v}$ with respect to the basis $\{ \partial/\partial x_i \rceil_{\pi(\boldsymbol{v})} \}_{i=1,\ldots,m}$ of $\mathscr{M}_{\pi(\boldsymbol{v})}$.

Thus $\tilde{\mathcal{F}} = \{(\tilde{U},\tilde{\phi}) | (U,\phi) \in \mathcal{F}\}$ is a differentiable structure on $T(\mathscr{M})$; the compatibility can be seen because if $\tilde{\phi}, \tilde{\psi} \in \tilde{\mathcal{F}}$ then the change of coordinates $\tilde{\psi} \circ \tilde{\phi}^{-1}$ involves the coordinate change in $\mathscr{M}$ and the Jacobean matrix of this change of coordinates: but both of them are C^∞ since we are on a C^∞ manifold, i.e. the differentiable structure is C^∞. Thus $T(\mathscr{M})$ is a topological space with a differentiable structure $\tilde{\mathcal{F}}$, i.e. it is a differentiable manifold.
The same strategy can be applied to all other cases.

□

From the definition we can see that vector bundles over a manifold are always locally trivial, in the sense that, given a coordinate system $(U,\phi) \in \mathcal{F}$, then

$$\pi^{-1}(U) \cong \phi(U) \times \mathbb{R}^m \cong U \times \mathbb{R}^m,$$

i.e. $\pi^{-1}(U)$ is a product $U \times \mathbb{R}^m$ up to diffeomorphisms.

Of course this property is in general not globally true. This motivates the following

Definition 3.24 (Parallelizable manifold)
A manifold is parallelizable *if $T(\mathscr{M})$ is a product $\mathscr{M} \times \mathbb{R}^d$ up to diffeomorphisms.*

3.8 Fields

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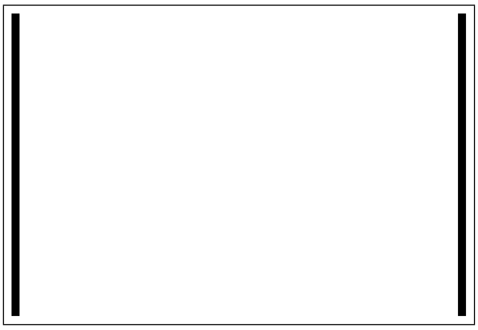

Figure 3.9: Local triviality.

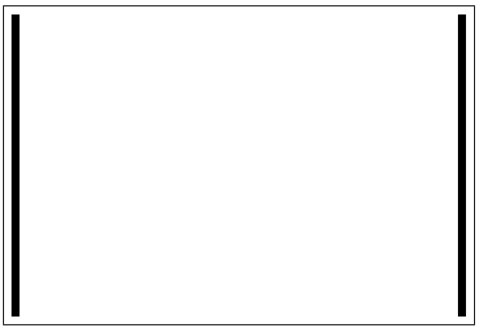

Figure 3.10: Parallelizable manifold.

Definition 3.25 (Smooth vector field)
A smooth vector field *over $\mathscr{M}$ is a section of $T(\mathscr{M})$. We will denote the space of all vector fields on $\mathscr{M}$ with $\mathcal{V}(\mathscr{M})$.*

Definition 3.26 (Line element field)
A line element field *over $\mathscr{M}$ is is a section of the line bundle over $\mathscr{M}$, i.e. it is a smooth assignment of a couple $(\boldsymbol{v}, -\boldsymbol{v})$ with $\boldsymbol{v} \in \mathscr{M}_{\mathtt{m}}$ at all $\mathtt{m} \in \mathscr{M}$.*

Proposition 3.7 (Characterization of vector fields)
Let $\boldsymbol{X}$ be a vector field on an open subset $W \subset \mathscr{M}$. The following properties are equivalent:

1. *$\boldsymbol{X}$ is differentiable;*
2. *given a chart $(U, \phi) \in \mathcal{F}$ with coordinate functions $(x_1, \ldots, x_m)$ if we consider*
$$\boldsymbol{X}\rceil_U = \sum_i^{1,m} a_i \frac{\partial}{\partial x_i},$$
then $a_i : U \subset \mathscr{M} \longrightarrow \mathbb{R}$ are differentiable functions on U;
3. *if $V \subset \mathscr{M}$ is open and $f \in C^\infty(V)$, then $\boldsymbol{X}(f) \in C^\infty(V)$, where we define*
$$\boldsymbol{X}(f)(\mathtt{m}) \stackrel{\text{def.}}{=} \boldsymbol{X}_{\mathtt{m}}(f)$$

Proof:

$1 \Rightarrow 2$ If $\boldsymbol{X}$ is differentiable then given a coordinate system (U, ϕ) then $\boldsymbol{X}\rceil_U$,
$$\boldsymbol{X}\rceil_U : U \longrightarrow T(\mathscr{M})$$
is differentiable. Moreover, since x_i is a coordinate function, $dx_i \circ \boldsymbol{X}\rceil_U$ is differentiable. But $dx_i \circ \boldsymbol{X}\rceil_U = a_i$ on U and the proof is complete.

$2 \Rightarrow 3$ On an open set V let us consider $f \in C^\infty(V)$. Let (U, ϕ) be a coordinate system on $\mathscr{M}$. Then
$$X(f) = \sum_i^{1,m} a_i \frac{\partial f}{\partial x_i}$$
is such that the a_i are differentiable functions by hypothesis and $\partial f / \partial x_i$ is differentiable since f is $C\infty$; thus $X(f)$ is also differentiable, as stated.

$3 \Rightarrow 1$ Let (U, ϕ) be a coordinate system on $\mathscr{M}$ chosen arbitrarily and let us call $(x_1, \ldots, x_m)$ the coordinate functions on U. Then
$$(x_1(\pi(\boldsymbol{v})), \ldots, x_1(\pi(\boldsymbol{v})), dx_1(\boldsymbol{v}), \ldots, dx_m(\boldsymbol{v}))$$
is a coordinate system on $T(\mathscr{M})$, i.e. it gives coordinates for each $\boldsymbol{v} \in \mathscr{M}_{\mathtt{m}}$ with $\mathtt{m} \in U$. Thus the differentiability of $x_i \circ \pi \circ X\rceil_U = x_i$ and of $dx_i \circ X\rceil_U = X(x_i)$ (which is implied by 3. with $f = x_i$) yields the differentiability of $X\rceil_U$.

The proof is so complete. $\square$

Definition 3.27 (Smooth 1-form field)
A smooth 1-form field *over $\mathscr{M}$ is a section of $T^*(\mathscr{M})$. We will denote the space of all smooth 1-form fields on $\mathscr{M}$ with $\mathcal{E}^1(\mathscr{M})$.*

Definition 3.28 (Smooth tensor field)
A smooth tensor field *over $\mathscr{M}$ is a section of $T_s^r(\mathscr{M})$. We will denote the space of all tensor fields of the (r, s) type on $\mathscr{M}$ with $\mathcal{T}_s^r(\mathscr{M})$.*

Proposition 3.8 (Characterization of smooth tensor fields)
Let $\boldsymbol{T}$ be a tensor field of the (r, s) type on an open subset $W \subset \mathscr{M}$. The following conditions are equivalent

Definition 3.29 (Smooth k-form field)
A smooth k-form field *over $\mathscr{M}$ is a section of $\Lambda^k(\mathscr{M})$. We will denote the space of all k-form fields on $\mathscr{M}$ with $\mathcal{E}^k(\mathscr{M})$.*

1. *$\boldsymbol{T}$ is differentiable;*
2. *given a chart $(U, \phi) \in \mathcal{F}$ with coordinate functions $(x_1, \ldots, x_m)$ if we consider*

$$\boldsymbol{T}\rceil_U = \sum_{\substack{i_1, \ldots, i_r \\ j_1, \ldots, j_s}}^{1,m} \boldsymbol{T}^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial x_{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \ldots \otimes dx_{j_s},$$

then $a^{i_1 \ldots i_r}_{j_1 \ldots j_s} : U \subset \mathscr{M} \longrightarrow \mathbb{R}$ are differentiable functions on U.

Proposition 3.9 (Characterization of smooth k-form fields)
Let $\boldsymbol{\omega}$ be a k-form field on an open subset $W \subset \mathscr{M}$. The following properties are equivalent:

1. *$\boldsymbol{\omega}$ is differentiable;*
2. *given a chart $(U, \phi) \in \mathcal{F}$ with coordinate functions $(x_1, \ldots, x_m)$ if we consider*

$$\boldsymbol{\omega}\rceil_U = \sum_{i_1 < \ldots < i_k}^{1,m} a_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

then $a_{i_1 \ldots i_k} : U \subset \mathscr{M} \longrightarrow \mathbb{R}$ are differentiable functions on U;
3. *if $V \subset \mathscr{M}$ is open and $V_1, \ldots, V_k$ are smooth vector fields on V, then then*

$$\boldsymbol{\omega}(V_1, \ldots, V_k)(m) \stackrel{\text{def.}}{=} \boldsymbol{\omega}(m)(V_1\rceil_m, \ldots, V_k\rceil_m)$$

Definition 3.30 (Smooth form field)
A smooth form field *over* $\mathscr{M}$ *is a section of* $\Lambda(\mathscr{M})$. *We will denote the space of all form fields on* $\mathscr{M}$ *with* $\mathcal{E}(\mathscr{M})$.

Proposition 3.10 (Characterization of smooth form fields)
Let $\boldsymbol{\omega}$ *be a form field on* $\mathscr{M}$. *Then* $\boldsymbol{\omega}$ *is smooth if and only if all the components in every given coordinate system* (U, ϕ) *are differentiable functions from* U *in* $\mathbb{R}$.

Definition 3.31 (Smooth vector field along a curve)
Let $\sigma(t)$ *be a curve on a manifold* $(\mathscr{M}, \mathcal{F})$. *A* smooth vector field along σ *is a differentiable map*

$$\boldsymbol{V} : [a, b] \subset \mathbb{R} \longrightarrow T(\mathscr{M})$$

such that

$$\pi \circ \boldsymbol{V}(t) = \sigma(t).$$

Analogous definitions can be given for form and tensor fields along a given curve σ.

Proposition 3.11 (Differential as a 1-form field)
Let $f : \mathscr{M} \longrightarrow \mathbb{R}$ *be a function on a differentiable manifold* $(\mathscr{M}, \mathcal{F}$. *The map*

$$df : \mathcal{M} \longrightarrow T^*(\mathscr{M}) = \Lambda^1(\mathscr{M})$$

defined as $df(m) = df\rceil_m$ *is a* 1*-form on* $\mathscr{M}$.

Proof:

To establish the result we have only to show that df is differentiable. This is true because of proposition 3.7, which can be applied if we consider a smooth vector field $\boldsymbol{V}$, remember that by definition $df(\boldsymbol{V}) = \boldsymbol{V}(f)$ and observe that $\boldsymbol{V}(f) : \mathscr{M} \longrightarrow \mathbb{R}$ is differentiable.
□

Proposition 3.12 (Existence of line element fields)
Every non-compact manifold admits a line element field.
A non-compact manifold admits a line element field if and only if its Euler invariant is zero.

3.9 Orientation on manifolds

Definition 3.32 (Orientation on a manifold)
A differentiable manifold $(\mathscr{M}, \mathcal{F})$ *of dimension* $\dim(\mathscr{M}) = m$ *is called* orientable *if there exists* $\mathcal{O} \subset \mathcal{F}$ *such that:*

1. $\{U_\alpha\}_{(U_\alpha, \phi_\alpha) \in \mathcal{O}}$ *is a cover of* $\mathcal{M}$*;*

2. *$\forall (U, \phi)$ and (V, ψ) coordinate systems of $\mathscr{M}$, with coordinate functions $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ respectively, it holds that the function*

$$\lambda : U \cap V \longrightarrow \mathbb{R}$$

defined by

$$dx_1 \wedge \ldots \wedge dx_m = \lambda dy_1 \wedge \ldots \wedge dy_m$$

is everywhere positive.

$\mathcal{O}$ is called an orientation *of $\mathscr{M}$.*

We know from the definition 1.10 that a choice of a basis in each tangent space gives an orientation of the tangent space itself. Moreover (proposition 1.15) change of basis of positive determinant select the same orientation in $\mathscr{M}_{\mathtt{m}}$. Thus the fact that in $U \cap V$ the function λ, which is nothing but the determinant of the change of coordinates $\psi \circ \phi^{-1}$, is always positive means that the two coordinate systems give the same orientation to the tangent spaces. Moreover a necessary and sufficient condition for a manifold to be orientable is given by the following proposition:

Proposition 3.13 (Characterization of orientable manifolds)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold. $\mathscr{M}$ is orientable if and only if there is a nowhere vanishing m-form on $\mathscr{M}$.

Proof:

$\Rightarrow$ Let $\mathcal{F}$ be the differentiable structure of $\mathscr{M}$ and let us assume $\mathscr{M}$ is orientable. Let $\mathcal{O}$ be an orientation. Let us consider the open cover

$$\mathcal{U} = \{\mathrm{supp}\,(\phi)\,|\phi \in \mathcal{O}\}$$

and a partition of unity $(\mathscr{R}, \mathcal{P})$ subordinated to $\mathscr{U}$. $\forall V \in \mathscr{R}$ let us consider a $\phi_V \in \mathcal{O}$ such that $V \subset \mathrm{supp}\,(\phi_V)$: we write the coordinate functions associated to this coordinate map as $x_1^{(V)}, \ldots, x_n^{(V)}$ and in this system of coordinates we can locally define an m-form

$$\boldsymbol{\omega}^{(V)} \stackrel{\text{def.}}{=} dx_1^{(V)} \wedge \ldots \wedge dx_m^{(V)}.$$

In terms of these m-forms we define

$$\boldsymbol{\omega} \stackrel{\text{def.}}{=} \sum_{V \in \mathscr{R}} f_V \boldsymbol{\omega}^{(V)},$$

where $f_V \in \mathcal{P}$. Now $\forall \mathtt{m} \in \mathscr{M}$ the set

$$\mathcal{A} = \{V \in \mathscr{R} | \mathtt{m} \in V\} = \{V_1, V_2, \ldots, V_n\}$$

is finite and around $\mathtt{m}$ we have

$$\begin{aligned} \boldsymbol{\omega} &= f_{V_1} \boldsymbol{\omega}^{(V_1)} + \ldots + f_{V_i} \boldsymbol{\omega}^{(V_i)} + \ldots + f_{V_n} \boldsymbol{\omega}^{(V_n)} \\ &= f_{V_1} dx_1^{(V_1)} \wedge \ldots \wedge dx_m^{(V_1)} + \ldots + \\ &\quad + f_{V_i} \det \left(\frac{\partial x_h^{(V_i)}}{\partial x_k^{(V_1)}} \right) dx_1^{(V_1)} \wedge \ldots \wedge dx_m^{(V_1)} + \ldots + \\ &\quad + f_{V_n} \det \left(\frac{\partial x_h^{(V_n)}}{\partial x_k^{(V_1)}} \right) dx_1^{(V_n)} \wedge \ldots \wedge dx_m^{(V_n)}. \end{aligned}$$

In the above written *local expression* all the determinants are positive, because the $V \in \mathscr{R}$ are subordinated to the orientation $\mathcal{O}$, all the f_V are non-negative, because of the definition of partition of unity, and at least one of them must be positive since again, by definition of partition of unity, their sum must equal 1. We thus obtain that $\forall \mathtt{m} \in \mathscr{M}$, $\boldsymbol{\omega}$ is non-vanishing at $\mathtt{m}$, i.e. $\boldsymbol{\omega}$ is a nowhere vanishing m-form on $\mathscr{M}$.
$\Leftarrow$ Let $\boldsymbol{\omega}$ be a nowhere vanishing m-form on $\mathscr{M}$. We ca without restriction assume that $\mathcal{F}$ is maximal. Then $\forall (U, \phi) \in \mathcal{F}$, if the coordinate functions associated to ϕ are $x_1^{(\phi)}$, ..., $x_m^{(\phi)}$, we can locally write

$$\boldsymbol{\omega} = f_\phi dx_1^{(\phi)} \wedge \ldots \wedge dx_m^{(\phi)},$$

where $f_\phi : U \longrightarrow \mathbb{R}$ is nowhere vanishing since ω satisfies this very property. Let us consider

$$\mathcal{O} \stackrel{\text{def.}}{=} \{(U, \phi) \in \mathcal{F} | f_\phi > 0\} :$$

since $\mathcal{F}$ is maximal, $\mathcal{O}$ is a cover of $\mathscr{M}$. Moreover if (U, ϕ) and (V, ψ) are two arbitrarily chosen coordinate systems of $\mathcal{O}$ with $U \cap V \neq 0$, then the change of coordinates is such that

$$dx_1^{(\psi)} \wedge \ldots \wedge dx_m^{(\psi)} = \frac{\boldsymbol{\omega}}{f_\psi} = \lambda dx_1^{(\phi)} \wedge \ldots \wedge dx_m^{(\phi)}$$

with $\lambda = f_\phi / f_\psi > 0$. Thus $\mathcal{O}$ defines an orientation of $\mathscr{M}$.

□

The nowhere vanishing m-form on $\mathscr{M}$ is called a *volume element* for $\mathscr{M}$.

Definition 3.33 (Regular domains and outer vectors)
Let $(\mathscr{M}, \mathcal{F})$ be an m-dimensional oriented manifold and let $D \subset \mathscr{M}$. D is called a regular domain if $\forall \mathtt{m} \in \mathscr{M}$ one of the following properties holds:

1. *$\exists U$ open neighborhood of $\mathtt{m}$ such that $U \subset \mathscr{M} - D$, i.e. $\mathtt{m}$ is* exterior *with respect to D;*
2. *$\exists U$ open neighborhood of $\mathtt{m}$ such that $U \subset D$, i.e. $\mathtt{m}$ is* interior *with respect to D;*
3. *$\exists (U, \phi)$ with $\mathtt{m} \in U$ and $\phi(\mathtt{m}) = (0, \ldots, 0)$ such that $\phi(U \cap D) = \phi(U) \cap \frac{1}{2}\mathbb{R}^m$, i.e. $\mathtt{m}$ is a* boundary *point.*

The union of all boundary points is ∂D, i.e. the boundary of D.
Let us now consider $\mathtt{m} \in \partial D$ and $\boldsymbol{v} \in \mathscr{M}_{\mathtt{m}}$. $\boldsymbol{v}$ is called an outer vector *with respect to D if there exists a smooth curve $\sigma(t)$ on $\mathscr{M}$ such that:*

1. *$\dot{\boldsymbol{\sigma}}(0) = \boldsymbol{v}$;*
2. *$\exists \epsilon > 0$ such that $\sigma(t) \notin D$ for $0 < t < \epsilon$.*

A maximal collection of coordinate systems as defined in *3.* above is compatible on ∂D and makes it differentiable manifold of dimension $m - 1$.

Definition 3.34 (Boundary of manifold)
Since $\mathscr{M}$ itself is a regular domain, $\partial \mathscr{M}$ is manifold of dimension $m-1$ (actually a submanifold of $\mathscr{M}$), which is called the boundary *of $\mathscr{M}$.*

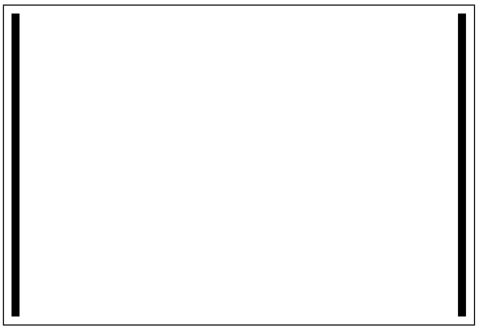

Figure 3.11: Induced orientation on the boundary.

With the concept of an outer vector we can now define a coherent way to give an orientation to a boundary.

Definition 3.35 (Orientation of the boundary)
Let $\mathscr{M}$ *be an orientable manifold,* D *a regular domain in* $\mathscr{M}$ *and* $\boldsymbol{v}$ *an outer vector with respect to* D*. Let us fix an orientation on* $\mathscr{M}$ *(i.e. fix coherently a basis in each* $\mathscr{M}_m$*) and then consider* $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n-1}$ *a basis of* $(\partial D)_m$*.* $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n-1}$ *is an* oriented basis *on* ∂D *if* $\boldsymbol{v}, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n-1}$ *gives the chosen orientation on* $\mathscr{M}$*.*
The choice of an oriented basis is an orientation *on* ∂D*.*

The definition given above is independent from the choice of the outer vector and gives a smooth orientation on ∂D in the sense of definition 3.32.

3.10 Exterior differential

Proposition 3.14 (Exterior differentiation)
Let $(\mathscr{M}, \mathcal{F})$ *be a differentiable manifold. There exists one and only one linear map*

$$d : \mathcal{E}^k(\mathscr{M}) \longrightarrow \mathcal{E}^{k+1}(\mathscr{M}) \quad \forall k \in \mathbb{N}$$

such that:

1. *if* f *is a differentiable function on* $\mathscr{M}$ *(i.e. a 0-form), then* $d(f) = df$*;*
2. *if* $\boldsymbol{\kappa}$ *is a* k*-form and* $\boldsymbol{\lambda}$ *an* l*-form over* $\mathscr{M}$*, then*
$$d(\boldsymbol{\kappa} \wedge \boldsymbol{\lambda}) = d\boldsymbol{\kappa} \wedge \boldsymbol{\lambda} + (-1)^k \boldsymbol{\kappa} \wedge d\boldsymbol{\lambda};$$
3. $d^2 = d \circ d = 0$.

3.11 Maps between manifolds

Definition 3.36 (Pullback)
Let $(\mathscr{M}, \mathcal{F})$ *and* $(\mathscr{N}, \mathcal{G})$ *be two differentiable manifolds and*

$$\psi : \mathscr{M} \longrightarrow \mathscr{N}$$

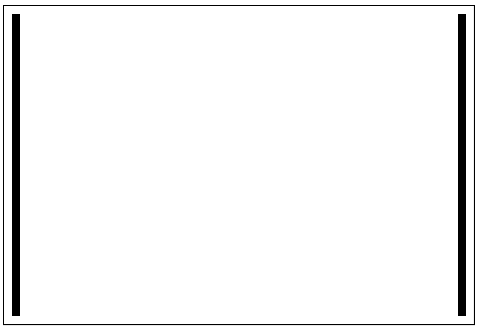

Figure 3.12: Pullback.

a differentiable map. ψ *induces in a natural way a map*

$$\psi_* : \mathcal{E}^k(\mathscr{N}) \longrightarrow \mathcal{E}^k(\mathscr{M}),$$

the pullback, *such that*

1. $\forall k \in \mathbb{N}$, $k > 0$, *given* $\omega \in \mathcal{E}^k(\mathscr{N})$ *then*

$$\psi_*(\boldsymbol{\omega})(m)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) \stackrel{\text{def.}}{=} \boldsymbol{\omega}(\psi(m))(d\psi\rceil_m(\boldsymbol{v}_1), \ldots, d\psi\rceil_m(\boldsymbol{v}_k));$$

2. *if* $k = 0$, *given* $f \in \mathcal{E}^0(\mathscr{N}) = C^\infty(\mathscr{N})$ *then*

$$\psi_*(f) \stackrel{\text{def.}}{=} f \circ \psi.$$

Proposition 3.15 (Properties of the pullback)
Let $(\mathscr{M}, \mathcal{F})$ *and* $(\mathscr{N}, \mathcal{G})$ *be two differentiable manifolds and*

$$\psi : \mathscr{M} \longrightarrow \mathscr{N}$$

a differentiable map; let ψ_* *be the associated pullback.*

1. *The pullback is linear, i.e.*

$$\psi_*(\omega \wedge \tau) = \psi_*(\omega) \wedge \psi_*(\tau).$$

2. *The pullback commutes with exterior differentiation, i.e.*

$$d \circ \psi_* = \psi_* \circ d.$$

3. *If* $(\mathscr{O}, \mathcal{H})$ *is a third manifold and*

$$\phi : \mathscr{N} \longrightarrow \mathscr{O}$$

then

$$(\phi \circ \psi)_* = \psi_* \circ \phi_*$$

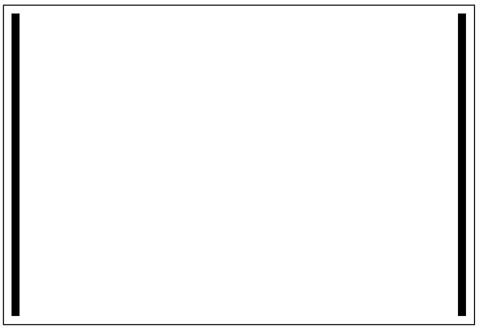

Figure 3.13: Pullback and exterior differentiation.

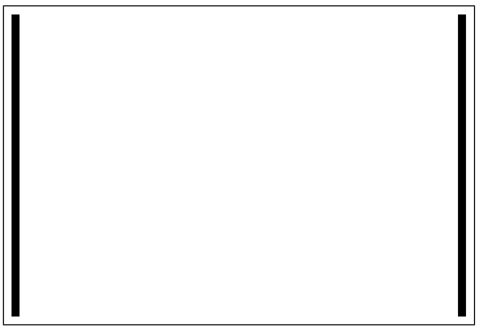

Figure 3.14: Pullback and composition.

A special important case is the one in which the map $\phi : \mathscr{M} \longrightarrow \mathscr{M}'$ is a diffeomorphism between two manifolds $\mathscr{M}$ and $\mathscr{M}'$. Then $\phi^{-1} : \mathscr{M}' \longrightarrow \mathscr{M}$ is again a diffeomorphism. ϕ_* maps forms of $\mathscr{M}'$ at $\phi(\mathtt{m})$ into forms of $\mathscr{M}$ at $\mathtt{m}$ and $(\phi^{-1})_*$ maps forms of $\mathscr{M}$ at $\mathtt{m}$ into forms of $\mathscr{M}'$ at $\phi(\mathtt{m})$. At the same time $d\phi$ maps vectors of $\mathscr{M}$ at minto vectors of $\mathscr{M}'$ at $\phi(\mathtt{m})$ and $d(\phi^{-1})$ maps vectors of $\mathscr{M}'$ at $\phi(\mathtt{m})$ into vectors of $\mathscr{M}$ at $\mathtt{m}$. Using the above relations and remembering the definition 3.16 we can induce in a canonical way a map between tensors.

Definition 3.37 (Tensor maps induced by diffeomorphisms)
Let $\mathscr{M}$ and $\mathscr{M}'$ be two differentiable manifolds and $\phi : \mathscr{M} \longrightarrow \mathscr{M}'$ be a diffeomorphism. Then ϕ induces a map $(\phi)^* : T^r_s(\mathscr{M}_m) \longrightarrow T^r_s(\mathscr{M}'_{\phi(m)})$, which associates to each (r,s)-tensor $\boldsymbol{T}$ at m an (r,s)-tensor $\phi^*\boldsymbol{T}$ at $\phi(m)$, according to the definition below

$$\begin{aligned}(\phi^*\boldsymbol{T})(\boldsymbol{\eta}'_1, \ldots, \boldsymbol{\eta}'_s, \boldsymbol{X}'_1, \ldots, \boldsymbol{X}'_r) &\overset{\text{def.}}{=} \\ &\overset{\text{def.}}{=} \boldsymbol{T}(\phi_*(\boldsymbol{\eta}'_1), \ldots, \phi_*(\boldsymbol{\eta}'_s), d\phi^{-1}(\boldsymbol{X}'_1), \ldots, d\phi^{-1}(\boldsymbol{X}'_r)) \\ &\qquad \forall \boldsymbol{\eta}'_1, \ldots, \boldsymbol{\eta}'_s \in (\mathscr{M}')^*_{\phi(m)}, \quad \forall \boldsymbol{X}'_1, \ldots, \boldsymbol{X}'_r \in \mathscr{M}'_{\phi(m)}. \end{aligned} \tag{3.4}$$

3.12 Vector fields and integral curves

Definition 3.38 (Lie Brackets)
Let us consider two vector fields $\boldsymbol{X}$, $\boldsymbol{Y} \in \mathscr{V}(\mathscr{M})$. The map

$$[-,-] : \mathscr{V}(\mathscr{M}) \times \mathscr{V}(\mathscr{M}) \longrightarrow \mathscr{V}(\mathscr{M})$$

which associates to $\boldsymbol{X}$ and $\boldsymbol{Y}$ the vector field $[\boldsymbol{X}, \boldsymbol{Y}]$ defined as

$$[\boldsymbol{X}, \boldsymbol{Y}]_m (f) \overset{\text{def.}}{=} \boldsymbol{X}_m(\boldsymbol{Y}(f)) - \boldsymbol{Y}_m(\boldsymbol{X}(f))$$

where

$$[\boldsymbol{X}, \boldsymbol{Y}]_m = [\boldsymbol{X}, \boldsymbol{Y}]\,(m) \in \mathscr{M}_m.$$

Proposition 3.16 (Properties of the Lie Brackets)
The Lie brackets have the following properties:

1. *$\forall f, g \in C^\infty(\mathscr{M})$ we have*

$$[f\boldsymbol{X}, g\boldsymbol{Y}] = fg\,[\boldsymbol{X}, \boldsymbol{Y}] + f\boldsymbol{X}(g)\boldsymbol{Y} - g\boldsymbol{Y}(f)\boldsymbol{X};$$

2. *it is antisymmetric, i.e. $[\boldsymbol{X}, \boldsymbol{Y}] = -[\boldsymbol{Y}, \boldsymbol{X}]$;*

3. *it satisfies the Jacobi identities, i.e.*

$$[[\boldsymbol{X}, \boldsymbol{Y}], \boldsymbol{Z}] + [[\boldsymbol{Y}, \boldsymbol{Z}], \boldsymbol{X}] + [[\boldsymbol{Z}, \boldsymbol{X}], \boldsymbol{Y}] = 0.$$

Definition 3.39 (Integral curves of a vector field)
Let $\boldsymbol{X} \in \mathscr{V}(\mathscr{M})$ be a smooth vector field on a manifold $(\mathscr{M}, \mathcal{F})$. A smooth curve σ on $\mathscr{M}$ is an integral curve of $\boldsymbol{X}$ if the tangent vector to the curve at all its points coincide with the values of the vector fields evaluated at those points, i.e.

$$\dot{\boldsymbol{\sigma}}(t) = \boldsymbol{X}_{\sigma(t)}.$$

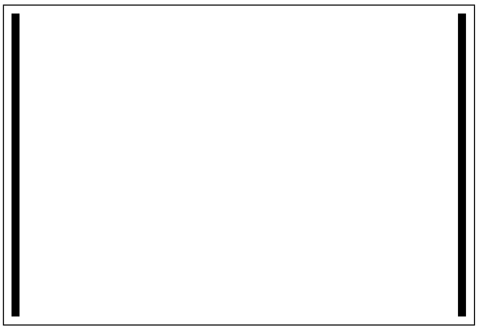

Figure 3.15: Integral curves of a vector field.

Proposition 3.17 (Equation satisfied by integral curves)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold and $\boldsymbol{V}$ a smooth vector field on $\mathscr{M}$. Let us consider $m \in U \subset \mathscr{M}$ with $(U, \phi) \in \mathcal{F}$ chart of $\mathscr{M}$ with coordinate functions $(x_1, \ldots, x_m)$ and let $\sigma(t)$ be a smooth curve on $\mathscr{M}$, such that $0 \in (a, b)$ and $\sigma(0) = m$. σ is an integral curve of $\boldsymbol{V}$ on U if and only if

$$\left.\frac{d\sigma_i}{dr}\right\rceil_t = v_i \circ \phi^{-1}(\sigma_1(t), \ldots, \sigma_m(t)) \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \quad t \in \phi^{-1}(U),$$

where we have

$$\boldsymbol{V}\rceil_U = \sum_i^{1,m} v_i \frac{\partial}{\partial x_i}$$

and

$$\sigma_i = x_i \circ \sigma.$$

Proof:

By definition 3.39, $\sigma(t)$ is an integral curve of the vector field $\boldsymbol{V}$ *if and only if* $\dot{\boldsymbol{\sigma}}(t) = \boldsymbol{V}_{\sigma(t)}$. But by definition 3.6 this is true *if and only if*

$$d\sigma\rceil_t \left(\left.\frac{d}{dr}\right\rceil_t \right) = \boldsymbol{V}_{\sigma(t)}. \tag{3.5}$$

By definition of differential we can write

$$d\sigma\rceil_t \left(\left.\frac{d}{dr}\right\rceil_t \right) = \sum_i^{1,m} \left.\frac{d(x_i \circ \sigma)}{dr}\right\rceil_t \left.\frac{\partial}{\partial x_i}\right\rceil_{\sigma(t)}$$

using the coordinate system (U, ϕ); moreover, always in (U, ϕ), the local expression of the vector field $\boldsymbol{V}$ along σ gives

$$\boldsymbol{V}_{\sigma(t)} = \sum_i^{1,d} v_i(\sigma(t)) \left.\frac{\partial}{\partial x_i}\right\rceil_{\sigma(t)}.$$

Substituting in (3.5) the last two equalities we thus get that $\sigma(t)$ is an integral curve of $\boldsymbol{V}$ on U *if and only if*

$$\sum_i^{1,m} \left.\frac{d(x_i \circ \sigma)}{dr}\right\rceil_t \left.\frac{\partial}{\partial x_i}\right\rceil_{\sigma(t)} = \sum_i^{1,d} v_i(\sigma(t)) \left.\frac{\partial}{\partial x_i}\right\rceil_{\sigma(t)},$$

i.e., since the $\partial/\partial x_i$ are linearly independent, *if and only if*

$$\left.\frac{d\sigma_i}{dr}\right]_t = v_i \circ \phi^{-1}(\sigma_1(t), \dots, \sigma_m(t)),\ i = 1, \dots, m,\ t \in \phi^{-1}(U).$$

□

Proposition 3.18 (Existence of integral curves)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold and $\boldsymbol{V}$ a smooth vector field on $\mathscr{M}$. $\forall \mathsf{m} \in \mathscr{M}$ there exists

a) a_{m} and $b_{\mathsf{m}} \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$;

b) a smooth curve $\sigma_{\mathsf{m}} : (a_{\mathsf{m}}, b_{\mathsf{m}}) \longrightarrow \mathscr{M}$

such that:

1. *$0 \in (a_{\mathsf{m}}), b_{\mathsf{m}})$ and $\sigma_{\mathsf{m}}(0) = \mathsf{m}$;*
2. *σ_{m} is an integral curve of $\boldsymbol{V}$;*
3. *if $\sigma' : (a', b') \longrightarrow \mathscr{M}$ is a smooth curve which satisfies a) and b) above, then $(a', b') \subseteq (a_{\mathsf{m}}, b_{\mathsf{m}})$ and $\sigma' = \sigma_{\mathsf{m}}\rceil_{(a',b')}$.*

   Moreover if $\forall t \in \mathbb{R}$ we define

   c) $\mathscr{D}_t = \{\mathsf{q} \in \mathscr{M} | t \in (a_{\mathsf{q}}, b_{\mathsf{q}})\} \subset \mathscr{M}$;
   d) $\Phi_t^{\boldsymbol{V}} : \mathscr{D}_t \subset \mathscr{M} \longrightarrow \mathscr{M}$ defined by $\Phi_t^{\boldsymbol{V}}(\mathsf{p}) = \sigma_{\mathsf{p}}(t)$;

   then

4. *$\forall \mathsf{m} \in \mathscr{M}$, there exists A open neighborhood of m and $\epsilon > 0$ such that*

   $$\Phi_{-}^{\boldsymbol{V}}(-) : (-\epsilon, +\epsilon) \times A \longrightarrow \mathscr{M}$$

   $\Phi_t^{\boldsymbol{V}}(\mathsf{p}) \stackrel{\text{def.}}{=} \sigma_{\mathsf{p}}(t)$, is smooth.
5. *$\mathscr{D}_t$ is open $\forall t \in \mathbb{R}$;*
6. *$\bigcup_{t>0} \mathscr{D}_t = \mathscr{M}$;*
7. *$\Phi_t^{\boldsymbol{V}} : \mathscr{D}_t \longrightarrow \mathscr{D}_{-t}$ is a diffeomorphism with inverse $\Phi_{-t}^{\boldsymbol{V}}$;*
8. *$\forall s, t \in \mathbb{R}$:*

   (a) $\operatorname{supp}\left(\Phi_s^{\boldsymbol{V}} \circ \Phi_t^{\boldsymbol{V}}\right) \subset \mathscr{D}_{s+t}$;
   (b) $\operatorname{supp}\left(\Phi_s^{\boldsymbol{V}} \circ \Phi_t^{\boldsymbol{V}}\right) = \mathscr{D}_{s+t}$ if $st > 0$;
   (c) in $\operatorname{supp}\left(\Phi_s^{\boldsymbol{V}} \circ \Phi_t^{\boldsymbol{V}}\right)$ we have $\Phi_s^{\boldsymbol{V}} \circ \Phi_t^{\boldsymbol{V}} = \Phi_{s+t}^{\boldsymbol{V}}$.

Definition 3.40 (Flow associated to a vector field)
The map $\Phi_{(-)}^{\boldsymbol{V}}(-) = \Phi^{\boldsymbol{V}}$ is called the flow *associated with the vector field $\boldsymbol{V}$.*

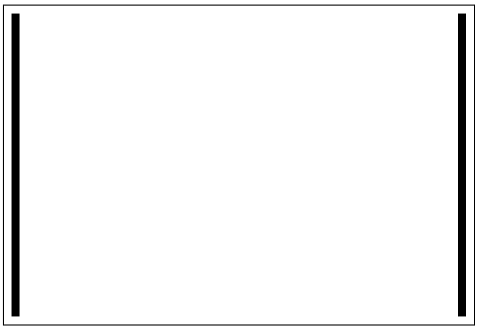

Figure 3.16: Flow associated with a vector field.

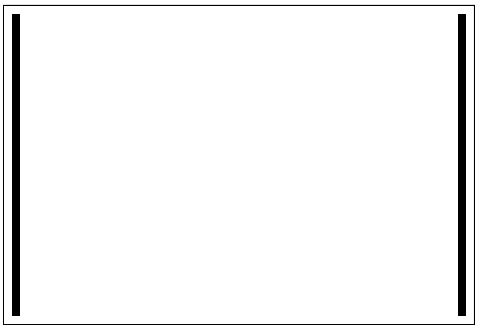

Figure 3.17: Lie derivative of a vector field.

3.13 Lie derivative

Definition 3.41 (Lie derivative of a vector field)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold and let $\boldsymbol{V}$ and $\boldsymbol{W}$ be two smooth vector fields on $\mathscr{M}$ and let $\Phi^{\boldsymbol{V}}$ be the flux associated with $\boldsymbol{V}$. Let us consider $m \in \mathscr{M}$ and

$$(\pounds_{\boldsymbol{V}} \boldsymbol{W})_m \stackrel{\text{def.}}{=} \lim_{t \to 0} \frac{d\Phi^{\boldsymbol{V}}_{-t}(\boldsymbol{Y}_{\Phi^{\boldsymbol{V}}_t(m)}) - \boldsymbol{Y}_m}{t} = \left.\frac{d}{dt}\right]_{t=0} (d\Phi^{\boldsymbol{V}}_{-t}(\boldsymbol{Y}_{\Phi^{\boldsymbol{V}}_t(m)})) \in \mathscr{M}_m.$$

$(\pounds_{\boldsymbol{V}} \boldsymbol{W})_m$ is the Lie derivative of $\boldsymbol{W}$ in the direction of $\boldsymbol{V}$ at m.

Definition 3.42 (Lie derivative of a 1-form field)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold, let $\boldsymbol{V}$ and $\boldsymbol{\omega}$ be a vector and a 1-form field on $\mathscr{M}$ respectively and let $\Phi^{\boldsymbol{V}}$ be the flux associated with $\boldsymbol{V}$. Let us consider $m \in \mathscr{M}$ and

$$(\pounds_{\boldsymbol{V}} \boldsymbol{\omega})_m \stackrel{\text{def.}}{=} \lim_{t \to 0} \frac{(\Phi^{\boldsymbol{V}}_t)_*(\boldsymbol{\omega}_{\Phi^{\boldsymbol{V}}_t(m)}) - \boldsymbol{\omega}_m}{t} = \left.\frac{d}{dt}\right]_{t=0} ((\Phi^{\boldsymbol{V}}_t)_*(\boldsymbol{\omega}_{\Phi^{\boldsymbol{V}}_t(m)})) \in \mathscr{M}_m.$$

$(\pounds_{\boldsymbol{V}} \boldsymbol{\omega})_m$ is the Lie derivative of $\boldsymbol{\omega}$ in the direction of $\boldsymbol{V}$ at m.

Proposition 3.19 (Properties of the Lie derivative)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold and $\boldsymbol{V}$ a smooth vector field on $\mathscr{M}$.

1. $\forall f \in C^\infty(\mathscr{M})$, $\pounds_{\boldsymbol{V}} f = \boldsymbol{V}(f)$.
2. $\forall \boldsymbol{W}$ *smooth vector field on* $\mathscr{M}$, $\pounds_{\boldsymbol{V}} \boldsymbol{W} = [\boldsymbol{V}, \boldsymbol{W}]$.
3. $\pounds_{\boldsymbol{V}}$ *maps smooth forms into smooth forms, i.e. can be considered as a map* $\pounds_{\boldsymbol{V}} : \mathcal{E}(\mathscr{M}) \longrightarrow \mathcal{E}(\mathscr{M})$; *moreover it commutes with exterior differentiation, i.e.* $\forall \boldsymbol{\omega} \in \mathcal{E}(\mathscr{M})$, $\pounds_{\boldsymbol{V}} d\boldsymbol{\omega} = d(\pounds_{\boldsymbol{V}} \boldsymbol{\omega})$ *and* $\pounds_{\boldsymbol{V}} = i(\boldsymbol{V}) \circ d + d \circ i(\boldsymbol{V})$.

The concept of Lie derivative can be extended to tensor of arbitrary rank, in which case it satisfies some further properties, as follows.

Proposition 3.20 (Lie derivative of arbitrary tensors)
The $\boldsymbol{S}$, $\boldsymbol{T}$ *be arbitrary tensors, Then the Lie derivatives satisfies the following properties:*

1. *it preserves tensor type;*
2. *it maps tensors linearly;*
3. *it preserves contraction;*
4. *it satisfies Leibnitz rule:*

$$\pounds_{\boldsymbol{V}} \boldsymbol{S} \otimes \boldsymbol{T} = \boldsymbol{S} \otimes \pounds_{\boldsymbol{V}} \boldsymbol{T} + \pounds_{\boldsymbol{V}} \boldsymbol{S} \otimes \boldsymbol{T}.$$

Moreover if $\boldsymbol{\omega}$ *is a k-form and* $\boldsymbol{W}_0$, $\boldsymbol{W}_1$, ..., $\boldsymbol{W}_k$ *are smooth vector fields, then*

5. $$\pounds_{\boldsymbol{W}_0}(\boldsymbol{\omega}(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_k)) = (\pounds_{\boldsymbol{W}_0} \boldsymbol{\omega})(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_k) + \\ + \sum_i^{1,k} \boldsymbol{\omega}(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_{i-1}, \pounds_{\boldsymbol{W}_0} \boldsymbol{W}_i, \boldsymbol{W}_{i+1}, \ldots, \boldsymbol{W}_k);$$

6. $$d\boldsymbol{\omega}(\boldsymbol{W}_0, \ldots, \boldsymbol{W}_k) = \sum_i^{0,k} (-1)^i \boldsymbol{W}_i(\boldsymbol{\omega}(\boldsymbol{W}_0, \ldots, \widehat{\boldsymbol{W}}_i, \ldots, \boldsymbol{W}_k)) + \\ + \sum_{i<j}^{0,k} (-1)^{i+j} \boldsymbol{\omega}([\boldsymbol{W}_i, \boldsymbol{W}_j], \ldots, \widehat{\boldsymbol{W}}_i, \ldots, \widehat{\boldsymbol{W}}_j, \ldots, \boldsymbol{W}_k).$$

Proposition 3.21 (Component expression of the Lie derivative)
The Lie Derivative of a vector field $\boldsymbol{Y}$ *in the direction of the vector field* $\boldsymbol{X}$ *can be expressed in a coordinate basis as*

$$(\pounds_{\boldsymbol{V}} \boldsymbol{Y})^i = \sum_j^{1,m} \left[\frac{\partial Y_i}{\partial x_j} V_j - \frac{\partial V_i}{\partial x_j} Y_j \right].$$

Proof:

We call $\mathtt{m}'$ a point along the integral curve of $\boldsymbol{V}$ passing through $\mathtt{m}$ at parameter distance t. Thus

$$\mathtt{m}' = \Phi_t^{\boldsymbol{V}}(\mathtt{m}).$$

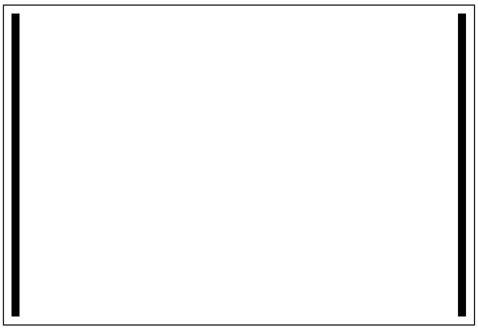

Figure 3.18: Component expression of the Lie derivative.

From the definition of the Lie derivative we see that we need to evaluate

$$\left.\frac{d}{dt}\right]_{t=0} (d\Phi^{\boldsymbol{V}}_{-t}(\boldsymbol{Y}_{\Phi^{\boldsymbol{V}}_t(\mathtt{m})}))$$

and we start with

$$(d\Phi^{\boldsymbol{V}}_{-t}(\boldsymbol{Y}_{\Phi^{\boldsymbol{V}}_t(\mathtt{m})}))^i$$

applying the definition of differential and of tangent vector. We get

$$\begin{aligned}
(d\Phi^{\boldsymbol{V}}_{-t}(\boldsymbol{Y}_{\Phi^{\boldsymbol{V}}_t(\mathtt{m})}))^i &= (d\Phi^{\boldsymbol{V}}_{-t}]_{\mathtt{m}'}(\boldsymbol{Y}_{\mathtt{m}'}))(x_i) \\
&= \boldsymbol{Y}_{\mathtt{m}'}\left(x_i \circ \Phi^{\boldsymbol{V}}_{-t}(\mathtt{m}')\right) \\
&= \sum_j^{1,m} (\boldsymbol{Y}_{\mathtt{m}'})_j \left.\frac{\partial}{\partial x_j}\right]_{\mathtt{m}'} \left(x_i \circ \Phi^{\boldsymbol{V}}_{-t}(\mathtt{m}')\right) \\
&= \sum_j^{1,m} Y_j(\mathtt{m}') \left.\frac{\partial x_i(\Phi^{\boldsymbol{V}}_{-t}(\mathtt{m}'))}{\partial x_j}\right]_{\mathtt{m}'}.
\end{aligned}$$

Of this expression we want to compute the derivative with respect to t and evaluate it at $t = 0$. There are two factors, we are going to use the chain rule and thus compute first the derivative of each factor, remembering that $\mathtt{m}'$ contains a t dependence as a function of m through he flux. We obtain in first place

$$\left.\frac{Y_j(\mathtt{m}')}{}\right]_{t=0} = \sum_k^{1,m} \left[\frac{\partial Y_j(\mathtt{m}')}{\partial x_k}\frac{dx_k}{dt}\right]_{t=0} = \sum_k^{1,m} \frac{\partial Y_j}{\partial x_k} X_k.$$

Moreover

$$\begin{aligned}
\left.\frac{d}{dt}\frac{\partial x_i(\Phi^{\boldsymbol{V}}_{-t}(\mathtt{m}'))}{\partial x_j}\right]_{t=0} &= \left.\frac{\partial}{\partial x_j}\;\frac{dx_i(\Phi^{\boldsymbol{V}}_{-t}(\mathtt{m}'))}{dt}\right]_{t=0} \\
&= \frac{\partial}{\partial x_j}(-X^i) \\
&= -\frac{\partial X_i}{\partial x_j}
\end{aligned}$$

Using the two results above we obtain

$$\left.\frac{d}{dt}\right]_{t=0} (d\Phi^{\boldsymbol{V}}_{-t}(\boldsymbol{Y}_{\Phi^{\boldsymbol{V}}_t(\mathtt{m})})) = \sum_j^{1,m} \left.\left[\frac{dY_j(\mathtt{m}')}{dt}\frac{\partial x_i(\Phi^{\boldsymbol{V}}_{-t}(\mathtt{m}'))}{\partial x_j}\right]_{\mathtt{m}'}\right]_{t=0} +$$

$$
\begin{aligned}
& + \sum_{j}^{1,m} \left[Y_j(\mathtt{m}') \frac{d}{dt} \frac{\partial x_i(\Phi^{\mathbf{V}}_{-t}(\mathtt{m}'))}{\partial x_j} \right|_{\mathtt{m}'} \Bigg]_{t=0} \\
= & \sum_{j}^{1,m} \sum_{k}^{1,m} \frac{\partial Y_j}{\partial x_k} X_k \delta_{ij} - \sum_{k}^{1,m} \frac{\partial X_i}{\partial x_k} Y_k \\
= & \sum_{k}^{1,m} \left[\frac{\partial Y_i}{\partial x_k} X_k - \frac{\partial X_i}{\partial x_k} Y_k \right],
\end{aligned}
$$

which completes the proof.

□

3.14 Integration on manifolds

Definition 3.43 (Integral of an m-form (local))
Let $(\mathscr{M}, \mathcal{F})$ be a manifold of dimension $\dim(\mathscr{M}) = m$ and let $(U, \phi) \in \mathcal{F}$ associated to the coordinate system $(x_1, \ldots, x_m)$. Let $\boldsymbol{\omega}$ be an m-form on $\mathscr{M}$ such that $\mathrm{supp}(\boldsymbol{\omega})$ is compact in $\mathscr{M}$. Then

$$\int_U \boldsymbol{\omega} \overset{\text{def.}}{=} \int_{\phi(U)} (\phi^{-1})_*(\boldsymbol{\omega}).$$

Proposition 3.22 (Local expression of the integral)
If in the local coordinates (U, ϕ) we write $\boldsymbol{\omega} = f_\phi \; dx_1 \wedge \ldots \wedge dx_m$ the following formula holds:

$$\int_{\phi(U)} (\phi^{-1})_*(\boldsymbol{\omega}) = \int_{\phi(U)} (f_\phi \circ \phi^{-1}) dr_1 \wedge \ldots \wedge dr_m.$$

Proof:

To prove the equality we have to evaluate $(\phi^{-1})_*(\boldsymbol{\omega})$ in the coordinate neighborhood (u, ϕ). This yields

$$
\begin{aligned}
(\phi^{-1})_*(\boldsymbol{\omega}) &= (\phi^{-1})_*(f_\phi dx_1 \wedge \ldots \wedge dx_m) \\
&= (\phi^{-1})_*(f_\phi) \cdot (\phi^{-1})_* dx_1 \wedge \ldots \wedge (\phi^{-1})_* dx_m \\
&= (f_\phi \circ \phi^{-1}) \cdot (\phi^{-1})_* dx_1 \wedge \ldots \wedge (\phi^{-1})_* dx_m \\
&= (f_\phi \circ \phi^{-1}) \cdot dr_1 \wedge \ldots \wedge dr_m,
\end{aligned}
$$

where the last equality holds since

$$
\begin{aligned}
(\phi^{-1})_* dx_i &= d((\phi^{-1})_* x_i) \\
&= d(x_i \circ \phi^{-1}) \\
&= d(r_i \circ \phi \circ \phi^{-1}) \\
&= dr_i.
\end{aligned}
$$

This completes the proof.

□

Proposition 3.23 (Coordinate independence of the local integral)
The definition 3.43 is independent from the choice of the coordinate neighborhood on $\mathscr{M}$*.*

Proof:

Let (U, ϕ) and (V, ψ) be two coordinate neighborhoods with coordinate functions $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ respectively. Let us consider the integral of an m-form $\boldsymbol{\omega}$ on some domain $W \subset U \cap V$. On W we have

$$\boldsymbol{\omega} = f_\phi dx_1 \wedge \ldots \wedge dx_m$$

as well as

$$\boldsymbol{\omega} = f_\psi dy_1 \wedge \ldots \wedge dy_m.$$

The change of coordinates is described by $\psi \circ \phi^{-1}$ which enters through its determinant $|\partial y_i / \partial x_j|$ in the relation between f_ψ and f_ϕ, since

$$f_\psi dy_1 \wedge \ldots \wedge dy_m = f_\psi \left| \frac{\partial y_i}{\partial x_j} \right| dx_1 \wedge \ldots \wedge dx_m,$$

so that

$$f_\phi = \left| \frac{\partial y_i}{\partial x_j} \right| f_\psi = |\phi \circ \psi^{-1}| f_\psi.$$

Moreover we remember the following property of the pullback

$$(\phi \circ \psi^{-1})_* = (\psi^{-1})_* \circ \phi_*.$$

Then

$$\begin{aligned}
\int_W \boldsymbol{\omega} &= \int_{\psi(W)} (\psi^{-1})_* \boldsymbol{\omega} \\
&= \int_{\psi(W)} (f_\psi \circ \psi^{-1}) d\rho_1 \wedge \ldots \wedge d\rho_m \\
&= \int_{(\phi\circ\psi^{-1})(\psi(W))} (f_\psi \circ \psi^{-1} \psi \circ \phi^{-1}) |\psi \circ \phi^{-1}| dr_1 \wedge \ldots \wedge dr_m \\
&= \int_{(\phi\circ\psi^{-1})(\psi(W))} (f_\psi \circ \psi^{-1} \psi \circ \phi^{-1}) |\psi \circ \phi^{-1}| dr_1 \wedge \ldots \wedge dr_m \\
&= \int_{\phi(W)} (|\psi \circ \phi^{-1}| f_\psi \circ \phi^{-1}) dr_1 \wedge \ldots \wedge dr_m \\
&= \int_{\phi(W)} (f_\phi \circ \phi^{-1}) dr_1 \wedge \ldots \wedge dr_m.
\end{aligned}$$

□

Definition 3.44 (Integral of an m-form (global))
Let $(\mathscr{M}, \mathcal{F})$ *be an orientable differentiable manifold of dimension* $\dim(\mathscr{M}) = m$ *and let* $\mathcal{O} \subset \mathcal{F}$ *be an orientation of* $\mathscr{M}$*. Let* $\boldsymbol{\omega}$ *be an* m*-form on* $\mathscr{M}$ *with* $\operatorname{supp}(\boldsymbol{\omega})$ *compact in* $\mathscr{M}$*. Let* $(\mathscr{R}, \mathcal{P})$ *be a partition of unity subordinated to the open covering* $\{U_\alpha\}_{(U_\alpha, \phi_\alpha) \in \mathcal{O}}$*. The integral of* $\boldsymbol{\omega}$ *over* $\mathscr{M}$ *is*

$$\int_{\mathscr{M}} \boldsymbol{\omega} \overset{\text{def.}}{=} \sum_{V \in \mathscr{R}} \int_V f_V \boldsymbol{\omega}.$$

Proposition 3.24 (Independence from the partition of unity choice)
The definition 3.44 is independent from the choice of the partition of unity.

Proof:

Let us take as granted all the assumptions in the definition 3.44 and let $(\mathscr{R}', \mathcal{P}')$ be another partition of unity subordinated to the open covering $\{U_\alpha\}_{(U_\alpha, \phi_\alpha) \in \mathcal{O}}$; then the following chain of equalities holds:

$$\begin{aligned} \sum_{V \in \mathscr{R}} \int_V f_V \boldsymbol{\omega} &= \sum_{V \in \mathscr{R}} \int_V \left(\sum_{V' \in \mathscr{R}'} f'_{V'} \right) f_V \boldsymbol{\omega} \\ &= \sum_{V \in \mathscr{R}} \sum_{V' \in \mathscr{R}'} \int_V f'_{V'} f_V \boldsymbol{\omega} \\ &= \sum_{\substack{V \in \mathscr{R} \\ V' \in \mathscr{R}'}} \int_{V \cap V'} f'_{V'} f_V \boldsymbol{\omega}. \end{aligned} \tag{3.6}$$

The last equality sign holds because f_V is different from zero only inside V and $f'_{V'}$ is different from zero only inside V', so the product $f'_{V'} f_V$ is different from zero inside $V' \cap V$. Since the last formula is symmetric we can proceed back with the chain of equalities, i.e.

$$\sum_{V \in \mathscr{R}} \int_V f_V \boldsymbol{\omega} = \sum_{V' \in \mathscr{R}'} \int_{V'} f'_{V'} \boldsymbol{\omega},$$

which completes the proof.

□

Proposition 3.25 (Stokes theorem)
Let $\mathscr{M}$ be an orientable differentiable manifold of dimension $\dim(\mathscr{M}) = m$ with boundary $\partial\mathscr{M}$. Let $\boldsymbol{\omega}$ be an $(m-1)$-form over $\mathscr{M}$. Then

$$\int_{\mathscr{M}} d\boldsymbol{\omega} = \int_{\partial\mathscr{M}} \boldsymbol{\omega}.$$

If $\mathscr{M}$ is such that $\partial\mathscr{M} = \emptyset$, then

$$\int_{\mathscr{M}} d\boldsymbol{\omega} = 0.$$

Since $\mathscr{M}$ is orientable, $\partial\mathscr{M}$ is also orientable and we assume on it the induced orientation (and of course the induced manifold structure (topological and differentiable)).

3.15 Riemannian and Lorentzian manifolds

Definition 3.45 (Riemannian metric)
Let us consider a manifold $(\mathscr{M}, \mathcal{F})$ and the set

$${}^{m}\langle \mathscr{M} \rangle = \bigcup_{m \in \mathscr{M}} \left\{ \langle -, - \rangle_{m} \,|\, \langle -, - \rangle_{m} \textit{ a positive definite metric on } \mathscr{M}_{m} \right\}.$$

A differentiable map

$$\langle -, - \rangle : \mathscr{M} \longrightarrow {}^{m}\langle \mathscr{M} \rangle,$$

defined as

$$\langle -, - \rangle (\mathtt{m}) \stackrel{\text{def.}}{=} \langle -, - \rangle_{\mathtt{m}},$$

is called a Riemannian metric *on* $\mathscr{M}$.

Differentiability is defined, as usual, in terms of vector fields, i.e. $\langle -, - \rangle$ is differentiable if for every choice of vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ on an open set $U \subset \mathscr{M}$ the function

$$\langle \boldsymbol{V}, \boldsymbol{W} \rangle : U \longrightarrow \mathbb{R},$$

defined as $\langle \boldsymbol{V}, \boldsymbol{W} \rangle (\mathtt{m}) \stackrel{\text{def.}}{=} \langle \boldsymbol{V}_{\mathtt{m}}, \boldsymbol{W}_{\mathtt{m}} \rangle_{\mathtt{m}}$, is differentiable.

Proposition 3.26 (Existence of Riemannian metric)
Every differentiable manifold admits a Riemannian metric.

Proof:

The proof of this statement proceeds along the same line we used for the characterization of the orientation on a manifold. Let thus $(\mathscr{M}, \mathcal{F})$ be a differentiable manifold, of dimension m. Let us consider a partition of unity $(\mathscr{R}, \mathcal{P})$ subordinated to the cover $\mathcal{U} = \{U | \ (U, \phi) \in \mathcal{F}\}$. For each $\mathtt{m} \in \mathscr{M}$, $\exists V \in \mathscr{R}$ such that $\mathtt{m} \in V$. Moreover $\exists U \in \mathcal{U}$ such that $V \subset U$ so that $\forall \mathtt{m} \in V \subset U$ in $\mathscr{M}_{\mathtt{m}}$ the coordinate map ϕ associated to U with coordinate functions $x_1, \ldots, x_m$ induces the coordinate basis $\{\partial / \, \partial x_i \rceil_{\mathtt{m}} \}_{i=1,\ldots,m}$. Thus $\forall \mathtt{m} \in V$ we can define a scalar product $\langle -, - \rangle_V$ by

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_V = \delta_{ij}.$$

Then

$$\langle -, - \rangle \stackrel{\text{def.}}{=} \sum_{V \in \mathscr{R}} f_V \, \langle -, - \rangle_V$$

is a Riemannian metric on $\mathscr{M}$.

□

Definition 3.46 (Lorentzian metric)
Let us consider a manifold $(\mathscr{M}, \mathcal{F})$ *and the set*

$${}^{m-2}\langle \mathscr{M} \rangle = \bigcup_{\mathtt{m} \in \mathscr{M}} \{ \langle -, - \rangle_{\mathtt{m}} | \, \langle -, - \rangle_{\mathtt{m}} \text{ a metric of signature } m - 2 \text{ on } \mathscr{M}_{\mathtt{m}} \} .$$

A differentiable map

$$\langle -, - \rangle : \mathscr{M} \longrightarrow {}^{m-2}\langle \mathscr{M} \rangle,$$

defined as

$$\langle -, - \rangle (\mathtt{m}) \stackrel{\text{def.}}{=} \langle -, - \rangle_{\mathtt{m}},$$

is called a Lorentzian metric *on* $\mathscr{M}$.

In what follows if we will refer to a metric, without specifying if it is Riemannian or Lorentzian, we will assume that the type of metric is not relevant, e.g. the results hold for the Riemannian as well as for the Lorentzian case.

Proposition 3.27 (Existence of Lorentzian metric)
A paracompact manifold admits a Lorentzian metric if and only if it admits a non-vanishing line element field.

Definition 3.47 (Isometry between manifolds)
Let $\mathscr{M}, \mathcal{F}$ and $\mathscr{N}, \mathcal{G}$ be two differentiable manifold and $\phi : \mathscr{M} \longrightarrow \mathscr{N}$ a map between them. ϕ is an isometry *if it is a diffeomorphism and if its differential $d\phi$ is a vector space isometry $\forall m \in \mathscr{M}$, i.e. if $\forall m \in \mathscr{M}$*

$$\langle d\phi(\boldsymbol{v}), d\phi(\boldsymbol{w})\rangle_{\phi(m)} = \langle \boldsymbol{v}, \boldsymbol{w}\rangle_{m} \quad , \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \mathscr{M}_m.$$

3.16 Connection and covariant derivative

Definition 3.48 (Connection at $\mathtt{m} \in \mathscr{M}$)
Let $\mathscr{M}$ be a differentiable manifold. A connection at $m \in \mathscr{M}$ *is a map*

$$D(-,-)_m : \mathscr{M}_m \times \mathcal{V}(\mathscr{M}) \longrightarrow \mathscr{M},$$

such that:

1. *$D(\boldsymbol{v}_m, \boldsymbol{W})_m$ is bilinear in $\boldsymbol{v}_m$ and W;*
2. *$\forall f : \mathscr{M} \longrightarrow \mathbb{R}$ differentiable,*

$$D(\boldsymbol{v}_m, f\boldsymbol{W})_m = \boldsymbol{v}_m(f)\boldsymbol{W}_m + f(m)D(\boldsymbol{v}_p, \boldsymbol{W}).$$

$D(\boldsymbol{v}_{\mathtt{m}}, \boldsymbol{W})_{\mathtt{m}}$ is called the *covariant derivative* of the vector field W in the direction of $\boldsymbol{v}_{\mathtt{m}}$ at $\mathtt{m}$.

Definition 3.49 (Connection on a manifold)
Let $\mathscr{M}$ be a differentiable manifold. A connection on $\mathscr{M}$ *is a map*

$$D(-,-) : \mathcal{V}(\mathscr{M}) \times \mathcal{V}(\mathscr{M}) \longrightarrow \mathcal{V}(\mathscr{M}),$$

such that:

1. *$D(\boldsymbol{V}, \boldsymbol{W})$ is bilinear in V and W;*
2. *$\forall f : \mathscr{M} \longrightarrow \mathbb{R}$ differentiable,*

$$D(f\boldsymbol{V}, \boldsymbol{W}) = fD(\boldsymbol{V}, \boldsymbol{W});$$

3. *$\forall f : \mathscr{M} \longrightarrow \mathbb{R}$ differentiable,*

$$D(\boldsymbol{V}, f\boldsymbol{W}) = \boldsymbol{V}(f)\boldsymbol{W} + fD(\boldsymbol{V}, \boldsymbol{W}).$$

We have that $\forall \mathtt{m} \in \mathscr{M}$

$$(D(\boldsymbol{V}, \boldsymbol{W}))_{\mathtt{m}} \stackrel{\text{def.}}{=} D(\boldsymbol{V}_{\mathtt{m}}, \boldsymbol{W})_{\mathtt{m}}$$

where $D(\boldsymbol{V}_{\mathtt{m}}, \boldsymbol{W})_{\mathtt{m}}$ is a connection at $\mathtt{m} \in \mathscr{M}$.

Definition 3.50 (Symmetric connection)
Let $\mathscr{M}, \mathcal{F}$ be a manifold and $D(-,-)$ a connection on $\mathscr{M}$. D is symmetric *if $\forall \boldsymbol{V}$, $\boldsymbol{W}$ vector fields on $\mathscr{M}$, then*

$$D(\boldsymbol{V}, \boldsymbol{W}) - D(\boldsymbol{W}, \boldsymbol{V}) = [\boldsymbol{V}, \boldsymbol{W}].$$

Definition 3.51 (Connection in coordinates)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold of dimension $\dim(\mathscr{M}) = m$ with connection $D(-,-)$ and let $(U, \phi) \in \mathcal{F}$ with coordinate functions $x_1, \ldots, x_m$. Then in the chart (U, ϕ) we have

$$D\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum_{k}^{1.m} \Gamma_{ij}^{k} \frac{\partial}{\partial x_k}$$

with

$$\Gamma_{ij}^{k} : U \longrightarrow \mathbb{R}$$

differentiable functions on $U \subset \mathscr{M}$.

Proposition 3.28 (Characterization of symmetric connections)
Let $D(-,-)$ be a connection on a manifold $\mathscr{M}, \mathcal{F}$ and $(U, \phi) \in \mathcal{F}$ a chart of $\mathscr{M}$ with coordinate functions $(x_1\ , \ \ldots\ , \ x_m)$. The following conditions are equivalent:

1. *$D(-,-)$ is symmetric;*
2. *$D\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right)$;*
3. *$\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$.*

Proof:

$1 \Rightarrow 2$ Let us consider a symmetric connection. In a coordinate basis of $\mathscr{M}_{\mathtt{m}}$, as is the one induced by the given chart, the Lie Brackets of two arbitrary basis vectors vanish, i.e.

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0.$$

Thus

$$D\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) - D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right) = 0$$

or

$$D\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right).$$

□

$2 \Rightarrow 3$ If we express

$$D\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right)$$

in terms of the connection symbols, the above equality becomes

$$\sum_{k}^{1,m} \left(\Gamma_{ij}^{k} - \Gamma_{ji}^{k}\right) \frac{\partial}{\partial x_k} = 0.$$

But, since $\{\partial/\partial x_k\}_{k=1,\ldots,m}$ is a basis of $\mathscr{M}_{\mathtt{m}}$ at each point $\mathtt{m} \in U \subset \mathscr{M}$, the $\partial/\partial x_k$ are linearly independent, i.e.

$$\Gamma_{ij}^{k} - \Gamma_{ji}^{k} = 0 \quad \Rightarrow \quad \Gamma_{ij}^{k} = \Gamma_{ji}^{k}.$$

□

3 ⇒ 1 We consider to arbitrary vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ and write them in a coordinate basis associated to a given chart (U, ϕ) with coordinate functions $x_1, \ldots, x_m$:

$$\begin{aligned} \boldsymbol{V} &= \sum_{i}^{1,m} v_i \frac{\partial}{\partial x_i} \\ \boldsymbol{W} &= \sum_{j}^{1,m} w_j \frac{\partial}{\partial x_j} \end{aligned} \tag{3.7}$$

We first compute

$$\begin{aligned} D(\boldsymbol{V}, \boldsymbol{W}) &= D\left(\sum_{i}^{1,m} v_i \frac{\partial}{\partial x_i}, \sum_{j}^{1,m} w_j \frac{\partial}{\partial x_j}\right) \\ &= \sum_{i,j}^{1,m} D\left(v_i \frac{\partial}{\partial x_i}, w_j \frac{\partial}{\partial x_j}\right) \\ &= \sum_{i,j}^{1,m} v_i D\left(\frac{\partial}{\partial x_i}, w_j \frac{\partial}{\partial x_j}\right) \\ &= \sum_{i,j}^{1,m} \left[v_i \frac{\partial w_j}{\partial x_i} \frac{\partial}{\partial x_j} + v_i w_j D\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right] \\ &= \sum_{i,j}^{1,m} v_i \frac{\partial w_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j,k}^{1,m} \Gamma_{ij}^{k} v_i w_j \frac{\partial}{\partial x_k}. \end{aligned}$$

Then, by exchanging $\boldsymbol{V}$ and $\boldsymbol{W}$ we also obtain

$$D(\boldsymbol{W}, \boldsymbol{V}) = \sum_{i,j}^{1,m} w_j \frac{\partial v_i}{\partial x_j} \frac{\partial}{\partial x_i} + \sum_{i,j,k}^{1,m} \Gamma_{ji}^{k} v_i w_j \frac{\partial}{\partial x_k},$$

so that

$$\begin{aligned} &D(\boldsymbol{V}, \boldsymbol{W}) - D(\boldsymbol{W}, \boldsymbol{V}) = \\ &\quad = \sum_{i,j}^{1,m} \left[v_i \frac{\partial w_j}{\partial x_i} \frac{\partial}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j} \frac{\partial}{\partial x_i}\right] + \\ &\qquad + \sum_{i,j,k}^{1,m} \left(\Gamma_{ji}^{k} - \Gamma_{ji}^{k}\right) v_i w_j \frac{\partial}{\partial x_k} \\ &\quad = \sum_{i,j}^{1,m} \left[v_i \frac{\partial w_j}{\partial x_i} \frac{\partial}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j} \frac{\partial}{\partial x_i}\right] \end{aligned} \tag{3.8}$$

since by the assumptions, $\Gamma^k_{ij} = \Gamma^k_{ji}$. We now compute the commutator, remembering in the first step result 1. of proposition 3.16:

$$\begin{aligned}
[\boldsymbol{V}, \boldsymbol{W}] &= \left[\sum_i^{1,m} v_i \frac{\partial}{\partial x_i}, \sum_j^{1,m} w_j \frac{\partial}{\partial x_j}\right] \\
&= \sum_{i,j}^{1,m} a_i b_j \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] + \\
&\quad + \sum_{i,j}^{1,m} v_i \frac{\partial w_j}{\partial x_i} \frac{\partial}{\partial x_j} - \sum_{i,j}^{1,m} w_j \frac{\partial v_i}{\partial x_j} \frac{\partial}{\partial x_i} \\
&= \sum_{i,j}^{1,m} \left[v_i \frac{\partial w_j}{\partial x_i} \frac{\partial}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j} \frac{\partial}{\partial x_i}\right] \qquad (3.9)
\end{aligned}$$

The first term in the equation before the last vanishes since we are in a coordinate basis and we thus see from (3.8) and (3.9) that

$$D(\boldsymbol{V}, \boldsymbol{W}) - D(\boldsymbol{W}, \boldsymbol{V}) = [\boldsymbol{V}, \boldsymbol{W}],$$

i.e. the connection is symmetric.

This completes the proof.

□

Proposition 3.29 (Covariant derivative along a curve)
Let $\sigma(t) : [a, b] \longrightarrow \mathscr{M}$ be a differentiable curve on a manifold $(\mathscr{M}, \mathcal{F})$ with connection $D(-,-)$. Let $\boldsymbol{V}(t)$ be a differentiable vector field along σ. There exists one and only one map which associates to a vector field $\boldsymbol{V}$ along σ another vector field $D\boldsymbol{V}/dt$ along σ, the covariant derivative of $\boldsymbol{V}$ along σ*, such that:*

1. $\dfrac{D(\boldsymbol{V}+\boldsymbol{W})}{dt} = \dfrac{D\boldsymbol{V}}{dt} + \dfrac{D\boldsymbol{W}}{dt}$;
2. $\forall f : [a, b] \longrightarrow \mathbb{R}$ *we have* $\dfrac{D(f\boldsymbol{V})}{dt} = \dfrac{df}{dt}\boldsymbol{V} + f\dfrac{D\boldsymbol{V}}{dt}$;
3. *if* $\boldsymbol{Y} \in \mathcal{V}(\mathscr{M})$ *is a vector field on* $\mathscr{M}$ *such that* $\boldsymbol{V}(t) = Y(\sigma(t))$ *then*

$$\frac{D\boldsymbol{V}}{dt} = D(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{Y})_{\sigma(t)}. \qquad (3.10)$$

Proof:

Let us choose a chart $(U, \phi) \in \mathcal{F}$ on the manifold $(\mathscr{M}, \mathcal{F})$ with coordinate functions $(x_1, \ldots, x_m)$. and let us consider a curve $\sigma(t) = (x_1(t), \ldots, x_m(t))$. We then have

$$\dot{\boldsymbol{\sigma}}(t) = \sum_i^{1,m} \frac{dx_i(t)}{dt} \frac{\partial}{\partial x_i}.$$

To prove the existence we use property 3. as an ansatz, i.e. we define

$$\frac{D\boldsymbol{V}}{dt} \stackrel{\text{def.}}{=} D\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{Y}\right)_{\sigma(t)};$$

this is a good definition since the operation defined by the connection is local, i.e. it depends only on the values of the vector fields at a given point and thus it makes sense for each vector field which is defined at that point. Moreover we have:

$$\begin{aligned}
\frac{D(\boldsymbol{V}+\boldsymbol{W})}{dt} &= D\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{V}+\boldsymbol{W}\right)_{\sigma(t)} \\
&= D\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{V}\right)_{\sigma(t)} + D\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{W}\right)_{\sigma(t)} \\
&= \frac{D\boldsymbol{V}}{dt} + \frac{D\boldsymbol{W}}{dt},
\end{aligned}$$

so that 1. is satisfied. Then we have

$$\begin{aligned}
\frac{D(f\boldsymbol{V})}{dt} &= D\left(\dot{\boldsymbol{\sigma}}(t), f\boldsymbol{V}\right)_{\sigma(t)} \\
&= (\dot{\boldsymbol{\sigma}}(t))(f)\boldsymbol{V} + fD\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{V}\right)_{\sigma(t)} \\
&= \sum_i^{1,m} \frac{dx_i(t)}{dt}\frac{\partial}{\partial x_i}(f) + fD\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{V}\right)_{\sigma(t)} \\
&= \frac{df}{dt}\boldsymbol{V} + f\frac{D\boldsymbol{V}}{dt}
\end{aligned}$$

and 2. is also satisfied. 3., of course, holds by definition, so the only property we still have to prove is uniqueness. To establish it we rewrite $D\boldsymbol{V}/dt$ using the local expression above for $\dot{\boldsymbol{\sigma}}(t)$ and also writing locally the vector field along $\sigma(t)$ as

$$\boldsymbol{V}(t) = \sum_j^{1,m} v_j(t)\frac{\partial}{\partial x_j}.$$

Then from equation (3.10) we can obtain the following chain of equalities:

$$\begin{aligned}
\frac{D\boldsymbol{V}}{dt} &= \frac{D\left(\sum_j^{1,m} v_j(t)(\partial/\partial x_j)\right)}{dt} \\
&= \sum_j^{1,m}\left(\frac{dv_j(t)}{dt}\frac{\partial}{\partial x_j} + v_j(t)\frac{D(\partial/\partial x_j)}{dt}\right) \\
&= \sum_j^{1,m}\left[\frac{dv_j(t)}{dt}\frac{\partial}{\partial x_j} + v_j(t)D\left(\dot{\boldsymbol{\sigma}}(t), \frac{\partial}{\partial x_j}\right)\right] \\
&= \sum_j^{1,m}\left[\frac{dv_j(t)}{dt}\frac{\partial}{\partial x_j} + v_j(t)D\left(\sum_i^{1,m}\frac{dx_i(t)}{dt}\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right] \\
&= \sum_k^{1,m}\frac{dv_k(t)}{dt}\frac{\partial}{\partial x_k} + \sum_{i,j}^{1,m} v_j(t)\frac{dx_i(t)}{dt}D\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \\
&= \sum_k^{1,m}\frac{dv_k(t)}{dt}\frac{\partial}{\partial x_k} + \sum_{i,j}^{1,m} v_j(t)\frac{dx_i(t)}{dt}\Gamma_{ij}^k\frac{\partial}{\partial x_k} \\
&= \sum_k^{1,m}\left(\frac{dv_k(t)}{dt} + \sum_{i,j}^{1,m}\Gamma_{ij}^k\frac{dx_i(t)}{dt}v_j(t)\right)\frac{\partial}{\partial x_k}. \qquad (3.11)
\end{aligned}$$

We thus see that the covariant derivative along $\sigma(t)$ is completely determined by the connection coefficients in a unique way, i.e., given the connection, it is unique. This completes the proof.

□

Definition 3.52 (Parallel vector field along a curve)
Let $\mathscr{M}, \mathscr{F}$ be a manifold with connection $D(-,-)$ and let $\sigma(t)$ be a curve on $\mathscr{M}$. A vector field $\boldsymbol{V}(t)$ along σ is parallel *along σ if*

$$\frac{D\boldsymbol{V}}{dt} = 0.$$

Proposition 3.30 (Characterization of parallel vector field)
Let $\mathscr{M}, \mathscr{F}$ be a manifold of dimension $\dim(\mathscr{M}) = m$ with connection $D(-,-)$. Let $(U, \phi) \in \mathcal{F}$ be a chart for $\mathscr{M}$ with coordinate functions $(x_1, \ldots, x_m)$ and let $\sigma(t) = (x_1(t), \ldots, x_m(t))$ be a curve on $\mathscr{M}$. A vector field $\boldsymbol{V}(t) = \sum_i^{1,m} v_i(t)\, \partial/\partial x_i$ along σ is parallel *along σ if and only if*

$$\frac{dv_k(t)}{dt} + \sum_{i,j}^{1,m} \frac{dx_i(t)}{dt} \Gamma^k_{ij} v_j(t) = 0 \qquad k = 1, \ldots, m. \tag{3.12}$$

Proposition 3.31 (Existence of parallel vector fields)
Let $\mathscr{M}, \mathscr{F}$ be a manifold and $\sigma(t) = (x_1(t), \ldots, x_m(t))$ be a curve on $\mathscr{M}$. Let $\boldsymbol{v}_0 \in \mathscr{M}_{\sigma(0)}$ be a tangent vector to $\mathscr{M}$ at $\sigma(0)$. There exists one and only one parallel vector field $\boldsymbol{V}$ along σ with $\boldsymbol{V}(\sigma(0)) = \boldsymbol{v}_0$.

Proposition 3.32 (Parallel translation is an isomorphism)
The parallel translation φ along a curve is an isomorphism

$$\varphi : \mathscr{M}_{\sigma(0)} \longrightarrow \mathscr{M}_{\sigma(t)} \quad , \quad \forall t \in [a, b].$$

3.17 Interplay between connection and metric

Definition 3.53 (Compatibility condition)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold with a metric $\langle -,- \rangle$. A connection $D(-,-)$ is compatible *with the metric $\langle -,- \rangle$ if $\forall \boldsymbol{V}\ \boldsymbol{W}$, parallel vector fields along an arbitrary given curve σ, it holds that $\langle \boldsymbol{V}, \boldsymbol{W} \rangle$ is constant along σ.*

This means that $\forall t$ for which σ is defined, the parallel translation along σ from $\sigma(0)$ to $\sigma(t)$ defines an isometry between $\mathscr{M}_{\sigma(0)}$ and $\mathscr{M}_{\sigma(t)}$.

Proposition 3.33 (Characterization of compatible connections: I)
A connection $D(-,-)$ on a manifold $(\mathscr{M}, \mathcal{F})$ with metric $\langle -,- \rangle$ is compatible with the metric if and only if $\forall \boldsymbol{V}$, $\boldsymbol{W}$, parallel vector fields along an arbitrary curve σ, the equality

$$\frac{d}{dt} \langle \boldsymbol{V}(t), \boldsymbol{W}(t) \rangle = \left\langle \frac{D\boldsymbol{V}}{dt}, \boldsymbol{W} \right\rangle + \left\langle \frac{D\boldsymbol{W}}{dt}, \boldsymbol{V} \right\rangle$$

is identically satisfied.

Proof:

$\Rightarrow$) Let us choose $\boldsymbol{P}_1, \ldots, \boldsymbol{P}_m$, m vector fields along σ which are orthonormal at a given point of σ. We can assume without restriction that they are parallel along σ (since given a vector at a point of a curve, to parallel propagate it along the curve we have only to solve the differential equations (3.11) = 0 with exactly the components of this vector as initial conditions). Then they are also orthonormal along σ, since their orthonormalization condition

$$\langle \boldsymbol{P}_i(t), \boldsymbol{P}_j(t) \rangle = \delta_{ij}$$

is preserved along σ precisely because the $\boldsymbol{P}_i$ are parallel along σ.

At every point of σ, we can thus write two arbitrary vector fields $\boldsymbol{V}$, $\boldsymbol{W}$ in terms of the orthonormal basis composed by the m vectors P_i, i.e.

$$\begin{aligned} \boldsymbol{V}(t) &= \sum_i^{1,m} v_i(t) \boldsymbol{P}_i(t) \\ \boldsymbol{W}(t) &= \sum_i^{1,m} w_i(t) \boldsymbol{P}_i(t). \end{aligned}$$

Moreover, since $\boldsymbol{P}_i$, $i = 1, \ldots, m$, are parallel vector fields, we also have

$$\begin{aligned} \frac{D\boldsymbol{V}(t)}{dt} &= \sum_i^{1,m} \frac{dv_i(t)}{dt} \boldsymbol{P}_i(t) + \sum_i^{1,m} v_i(t) \frac{D\boldsymbol{P}_i(t)}{dt} \\ &= \sum_i^{1,m} \frac{dv_i(t)}{dt} \boldsymbol{P}_i(t) \\ \frac{D\boldsymbol{W}(t)}{dt} &= \sum_j^{1,m} \frac{dw_j(t)}{dt} \boldsymbol{P}_j(t) + \sum_j^{1,m} w_j(t) \frac{D\boldsymbol{P}_j(t)}{dt} \\ &= \sum_j^{1,m} \frac{dw_j(t)}{dt} \boldsymbol{P}_j(t). \end{aligned}$$

Remembering that the $\boldsymbol{P}_i(t)$ are orthonormal along σ, we can now compute

$$\begin{aligned} \left\langle \frac{D\boldsymbol{V}}{dt}, \boldsymbol{W} \right\rangle &= \left\langle \sum_i^{1,m} \frac{dv_i(t)}{dt} \boldsymbol{P}_i(t), \sum_j^{1,m} w_j(t) \boldsymbol{P}_j(t) \right\rangle \\ &= \sum_{i,j}^{1,m} \frac{dv_i(t)}{dt} w_j(t) \langle \boldsymbol{P}_i(t), \boldsymbol{P}_j(t) \rangle \\ &= \sum_{i,j}^{1,m} \frac{dv_i(t)}{dt} w_j(t) \delta_{ij} \\ &= \sum_i^{1,m} \frac{dv_i(t)}{dt} w_i(t) \end{aligned}$$

and exchanging $\boldsymbol{V}$ with $\boldsymbol{W}$

$$\left\langle \boldsymbol{V}, \frac{D\boldsymbol{W}}{dt} \right\rangle = \sum_i^{1,m} \frac{dw_i(t)}{dt} v_i(t);$$

Summing the last two result we thus get

$$\begin{aligned}\left\langle \frac{D\boldsymbol{V}}{dt}, \boldsymbol{W}\right\rangle + \left\langle \boldsymbol{V}, \frac{D\boldsymbol{W}}{dt}\right\rangle &= \sum_i^{1,m}\left(\frac{dv_i(t)}{dt}w_i(t) + \frac{dw_i(t)}{dt}v_i(t)\right)\\ &= \frac{d}{dt}\left(\sum_i^{1,m} v_i w_i\right)\\ &= \frac{d}{dt}\left\langle \boldsymbol{V}, \boldsymbol{W}\right\rangle,\end{aligned}$$

which completes the proof of this implication.
$\Leftarrow$) If $\boldsymbol{V}$ and $\boldsymbol{W}$ are parallel along σ then $D\boldsymbol{V}/dt = D\boldsymbol{W}/dt = 0$, i.e.

$$\frac{d}{dt}\left\langle \boldsymbol{V}(t), \boldsymbol{W}(t)\right\rangle = 0$$

so that

$$\left\langle \boldsymbol{V}(t), \boldsymbol{W}(t)\right\rangle = \text{const.}$$

and D is compatible with the metric.

□

Proposition 3.34 (Characterization of compatible connections: II) *A connection $D(-,-)$ on a manifold $(\mathscr{M}, \mathcal{F})$ with metric $\langle -,-\rangle$ is compatible with the metric if and only if $\forall \boldsymbol{V}$, $\boldsymbol{W}$, $\boldsymbol{Z}$ vector fields on $\mathscr{M}$ it holds that*

$$\boldsymbol{Z}(\langle \boldsymbol{V}, \boldsymbol{W}\rangle) = \langle D(\boldsymbol{Z}, \boldsymbol{V}), \boldsymbol{W}\rangle + \langle D(\boldsymbol{Z}, \boldsymbol{W}), \boldsymbol{V}\rangle. \tag{3.13}$$

Proof:

Let σ be a differentiable curve on $\mathscr{M}$ such that

$$\begin{aligned}\sigma(0) &= \mathtt{m} \in \mathscr{M}\\ \dot{\boldsymbol{\sigma}}(0) &= \boldsymbol{Z}_{\mathtt{m}} \in \mathscr{M}_{\mathtt{m}}.\end{aligned}$$

Remembering these settings we preliminarily define the following quantities:

$$\boldsymbol{Z}_{\mathtt{m}}(\langle \boldsymbol{V}, \boldsymbol{W}\rangle) \tag{3.14}$$

$$\left.\tfrac{d}{dt}\right\rceil_{t=0}\left\langle \boldsymbol{V}_{\sigma(t)}, \boldsymbol{W}_{\sigma(t)}\right\rangle \tag{3.15}$$

$$\left\langle \left.\tfrac{D\boldsymbol{V}}{dt}\right\rceil_{t=0}, \boldsymbol{W}\right\rangle + \left\langle \left.\tfrac{D\boldsymbol{W}}{dt}\right\rceil_{t=0}, \boldsymbol{V}\right\rangle \tag{3.16}$$

$$\langle D(\boldsymbol{Z}_{\mathtt{m}}, \boldsymbol{V}), \boldsymbol{W}\rangle + \langle D(\boldsymbol{Z}_{\mathtt{m}}, \boldsymbol{W}), \boldsymbol{V}\rangle \tag{3.17}$$

$\Rightarrow$) We now start with the direct implication. We can compute the directional derivative of the function $\langle \boldsymbol{U}, \boldsymbol{V}\rangle$ in the direction of $\boldsymbol{W}_{\mathtt{m}}$ as the derivative along the curve σ at $t = 0$: since this directional derivative is a local expression it does not depend on the chosen curve, provided it has tangent vector $\boldsymbol{W}_{\mathtt{m}}$ at $\mathtt{m}$. This says (3.14) = (3.15). Using proposition 3.33 we know that (3.15) = (3.16) and by definition of the covariant derivative along a curve (3.16) = (3.17). Thus (3.14) = (3.17) $\forall \mathtt{m} \in \mathscr{M}$, which is equivalent to (3.13), the result to be established.
$\Leftarrow$) To prove the converse we observe that now (3.13) holds, so that (3.14) = (3.17) $\forall \mathtt{m} \in \mathscr{M}$. But again, by the same considerations we made

above, (3.14) = (3.15) and (3.17) = (3.16). So we have established that under the assumed conditions (3.15) = (3.16) i.e. that $\forall \boldsymbol{V}, \boldsymbol{W}$ vector fields along a curve σ

$$\frac{d}{dt}\langle \boldsymbol{V}, \boldsymbol{W}\rangle = \left\langle \frac{D\boldsymbol{V}}{dt}, \boldsymbol{W}\right\rangle + \left\langle \frac{D\boldsymbol{W}}{dt}, \boldsymbol{V}\right\rangle$$

so that proposition 3.33 assures the connection is compatible.

□

Proposition 3.35 (Uniqueness of symmetric compatible connection) *Let $(\mathscr{M}, \mathcal{F})$ be a manifold and $\langle -, - \rangle$ a metric on $\mathscr{M}$. There exists one and only one symmetric connection on $\mathscr{M}$ compatible with the given metric.*

Proof:

We will prove the uniqueness: let $(U, \phi) \in \mathcal{F}$ be a coordinate system with coordinate functions $x_1, \ldots, x_m$. As usual, we have that locally the connection can be expressed as

$$D\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right) = \sum_k^{1,m} \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

and the Riemannian metric as

$$g_{mn} = \left\langle \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_n}\right\rangle.$$

The compatibility condition implies

$$\begin{aligned} \frac{\partial}{\partial x_j}\left\langle \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right\rangle &= \left\langle D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right), \frac{\partial}{\partial x_l}\right\rangle + \\ &\quad + \left\langle D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l}\right), \frac{\partial}{\partial x_k}\right\rangle \end{aligned} \tag{3.18}$$

and permuting the indices j, k and l we also get

$$\begin{aligned} \frac{\partial}{\partial x_k}\left\langle \frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_j}\right\rangle &= \left\langle D\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right), \frac{\partial}{\partial x_j}\right\rangle + \\ &\quad + \left\langle D\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}\right), \frac{\partial}{\partial x_l}\right\rangle \end{aligned} \tag{3.19}$$

$$\begin{aligned} \frac{\partial}{\partial x_l}\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle &= \left\langle D\left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_j}\right), \frac{\partial}{\partial x_k}\right\rangle + \\ &\quad + \left\langle D\left(\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_k}\right), \frac{\partial}{\partial x_j}\right\rangle \\ &= \left\langle D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l}\right), \frac{\partial}{\partial x_k}\right\rangle + \\ &\quad + \left\langle D\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right), \frac{\partial}{\partial x_j}\right\rangle, \end{aligned} \tag{3.20}$$

where in the last equality we have used the fact that the connection is symmetric and the arguments are element of a coordinate basis, so that result 2. of proposition 3.28 applies. Summing side by side (3.18) and (3.19) and subtracting (3.20) we get

$$\begin{aligned}
&\frac{\partial}{\partial x_j}\left\langle \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right\rangle + \frac{\partial}{\partial x_k}\left\langle \frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_j}\right\rangle - \frac{\partial}{\partial x_l}\left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right\rangle = \\
&\quad = \left\langle D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right), \frac{\partial}{\partial x_l}\right\rangle + \left\langle D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l}\right), \frac{\partial}{\partial x_k}\right\rangle + \\
&\quad\quad + \left\langle D\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right), \frac{\partial}{\partial x_j}\right\rangle + \left\langle D\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}\right), \frac{\partial}{\partial x_l}\right\rangle + \\
&\quad\quad - \left\langle D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l}\right), \frac{\partial}{\partial x_k}\right\rangle - \left\langle D\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right), \frac{\partial}{\partial x_j}\right\rangle \\
&\quad = \left\langle D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right), \frac{\partial}{\partial x_l}\right\rangle + \left\langle D\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}\right), \frac{\partial}{\partial x_l}\right\rangle \\
&\quad = 2\left\langle D\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right), \frac{\partial}{\partial x_l}\right\rangle,
\end{aligned}$$

where in the last line we again used the symmetry property of the connection.
The equality coming from the first and last lines can be rewritten, using the metric the and connection symbols in the chosen coordinate system that we have written at the beginning of this proof, as

$$-\partial_l g_{jk} + \partial_j g_{kl} + \partial_k g_{lj} = 2\sum_h^{1,m} \Gamma^h_{jk} g_{hl}$$

or, acting with the inverse of the metric[1], as

$$\Gamma^i_{jk} = \frac{1}{2}\sum_l^{1,m} (g^{-1})_{il}\left(-\partial_l g_{jk} + \partial_j g_{kl} + \partial_k g_{lj}\right) \tag{3.21}$$

... To prove the existence we use the definition above and verify that all properties of a connection are satisfied.

□

Notation 3.1 (Compatible Symmetric Covariant Derivative)
When we consider the unique symmetric connection compatible with a metric on a manifold, we are going to use the following notation:

$$\boldsymbol{\nabla}_{\boldsymbol{V}} \boldsymbol{W} = D\left(\boldsymbol{V}, \boldsymbol{W}\right).$$

3.18 Geodesics

Definition 3.54 (Geodesic)
Let σ be a curve on a manifold $(\mathscr{M}, \mathcal{F})$ such that the vector field $\dot{\boldsymbol{\sigma}}(t)$ tangent

[1] Here we will denote the inverse metric tensor as $(g^{-1})_{ik}$, i.e.

$$\sum_k^{1,m} (g^{-1})_{ik} g_{kj} = \sum_k^{1,m} g_{ik}(g^{-1})_{kj} = \delta_{ij}.$$

Of course we have $g_{ij} = g_{ji}$ and $(g^{-1})_{ij} = (g^{-1})_{ji}$.

to the curve is parallel along the curve, i.e.

$$\frac{D\dot{\boldsymbol{\sigma}}(t)}{dt} = 0.$$

Then σ is a geodesic *on $\mathscr{M}$.*

Proposition 3.36 (Geodesic equation)
Let σ be a geodesic on $(\mathscr{M}, \mathcal{F})$ and let $(U, \phi) \in \mathcal{F}$ be a chart of $\mathscr{M}$ with coordinate functions $(x_1, \ldots, x_m)$. σ is a geodesic if and only if

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j}^{1,m} \Gamma^k_{ij} \frac{dx_i(t)}{dt} \frac{dx_j(t)}{dt} = 0, \quad k = 1, \ldots, m \tag{3.22}$$

where $\phi \circ \sigma(t) = (x_1(t), \ldots, x_m(t))$.

Proof:

In the given coordinates the tangent vector $\dot{\boldsymbol{\sigma}}$ has components:

$$\left(\frac{dx_1(t)}{dt}, \ldots, \frac{dx_m(t)}{dt}\right).$$

But, by the definition above, σ is a geodesic if and only if this tangent vector is a parallel vector field along σ and this is true if and only if it satisfies the m differential equations (3.12) of proposition 3.30, where now $v_l = dx_l(t)/dt$. Thus a curve σ is a geodesic if and only if locally the m equations (3.22) are satisfied.

□

Definition 3.55 *Let $(\mathscr{M}, \mathcal{F})$ be a manifold and let $\mathtt{m} \in \mathscr{M}$ and $\boldsymbol{v} \in \mathscr{M}_{\mathtt{m}}$ be a point of $\mathscr{M}$ and a vector tangent to $\mathscr{M}$ at $\mathtt{m}$ respectively. The* exponentiation of $\boldsymbol{v}$ at $\mathtt{m}$ *is the point $\mathtt{p} \in \mathscr{M}$ which is a unit parameter distance away along the unique geodesic $\sigma_{\boldsymbol{v}}$ passing at $\mathtt{m}$ at $t = 0$ and having at $\mathtt{m}$ velocity $\boldsymbol{v}$. The* exponential *of the vector $\boldsymbol{v}$ at $\mathtt{m}$ is thus defined as*

$$\exp_{\mathtt{m}}(\boldsymbol{v}) \stackrel{\text{def.}}{=} \sigma_{\boldsymbol{v}}(1)$$

and is a map

$$\exp_{\mathtt{m}} : W \subset \mathscr{M}_{\mathtt{m}} \longrightarrow \mathscr{M}.$$

If k is a constant then of course we have

$$\left.\frac{d}{dt}\right|_{t=0} \sigma_{\boldsymbol{v}}(kt) = k\boldsymbol{v}$$

so that

$$\exp_{\mathtt{m}}(k\boldsymbol{v}) = \sigma_{\boldsymbol{v}}(k).$$

If we interpret k as a parameter we thus have that

$$\sigma_{\boldsymbol{v}}(t) \stackrel{\text{def.}}{=} \exp_{\mathtt{m}}(t\boldsymbol{v})$$

is the only geodesics with $\sigma(0) = \mathtt{m}$ and $\dot{\boldsymbol{\sigma}}(0) = \boldsymbol{v}$.

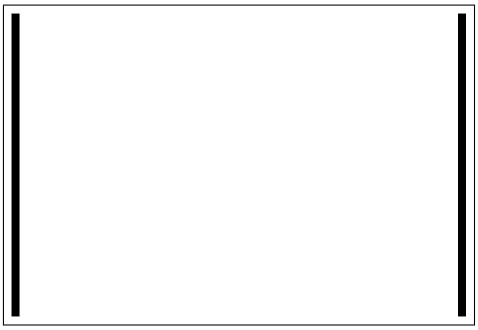

Figure 3.19: Exponential of a vector.

Proposition 3.37 *The differential of the exponential map at* $\mathtt{m}$ *is an isomorphism of* $\mathscr{M}_{\mathtt{m}}$*, in particular*

$$d(\exp_{\mathtt{m}})\rceil_{\mathbf{0}} = \mathbb{I}_{\mathscr{M}_{\mathtt{m}}}.$$

Proof:

To understand $d(\exp_{\mathtt{m}})$ let us remember that according to its definition, the differential is a map between tangent spaces. In this case the tangent spaces are:

1. the tangent space to the tangent space $\mathscr{M}_{\mathtt{m}}$ at the origin $\mathtt{0}$, which we could denote with $(\mathscr{M}_{\mathtt{m}})_{\mathtt{0}}$; since $\mathscr{M}_{\mathtt{m}}$ is a vector space, we can up to an isomorphism use the following identification
$$(\mathscr{M}_{\mathtt{m}})_{\mathtt{0}} \approx \mathscr{M}_{\mathtt{m}};$$
2. the tangent space to $\mathscr{M}$ at $\mathtt{m}$, i.e. $\mathscr{M}_{\mathtt{m}}$.

Thus up to an isomorphism

$$d(\exp_{\mathtt{m}}) : \mathscr{M}_{\mathtt{m}} \longrightarrow \mathscr{M}_{\mathtt{m}}.$$

We are now interested in the action of $d(\exp_{\mathtt{m}})$ on $\mathscr{M}_{\mathtt{m}}$. To grasp it we can take a curve in $\mathscr{M}_{\mathtt{m}}$ which has $\boldsymbol{v}$ as tangent vector, consider its image under $\exp_{\mathtt{m}}$ and look for the tangent vector of this image (which is a curve on $\mathscr{M}$).
As a curve in $\mathscr{M}_{\mathtt{m}}$ with tangent vector $\boldsymbol{v}$ at the origin $\mathtt{0}$ we can choose the line $l(t) = t\boldsymbol{v}$, $-\epsilon < t < +\epsilon$, for some small enough $\epsilon > 0$. Of course

$$\left.\frac{d}{dt}\right\rceil_{\mathtt{0}} l(t) = \boldsymbol{v}.$$

The exponential then maps this curve into the unique geodesic $\gamma(t) = \exp_{\mathtt{m}}(t\boldsymbol{v})$ passing through $\mathtt{m}$ with tangent vector

$$\left.\frac{d}{dt}\right\rceil_{\mathtt{0}} \gamma(t) = \boldsymbol{v},$$

since $\exp_{\mathtt{m}}$ is defined exactly in this way. Thus

$$d(\exp_{\mathtt{m}})\rceil_{\mathtt{0}} \left(\left.\frac{d}{dt}\right\rceil_{\mathtt{0}} l(t) \right) = \left.\frac{d}{dt}\right\rceil_{\mathtt{0}} \gamma(t) \quad \Rightarrow \quad d(\exp_{\mathtt{m}})\rceil_{\mathtt{0}} (\boldsymbol{v}) = \boldsymbol{v}$$

and this holds $\forall \boldsymbol{v} \in \mathscr{M}_{\mathtt{m}}$, i.e.

$$d(\exp_{\mathtt{m}})\rceil_{\mathbf{0}} = \mathbb{I}_{\mathscr{M}_{\mathtt{m}}}.$$

$\square$

From the above results and the implicit function theorem we conclude that the exponential map is a local diffeomorphism around $\mathbf{0} \in \mathscr{M}_{\mathtt{m}}$ onto a neighborhood $U \subset \mathscr{M}$ of $\mathtt{m}$. It maps lines in the tangent space in geodesics of $\mathscr{M}$ passing through $\mathtt{m}$ and having tangent vector at $\mathtt{m}$ which is the director vector of the line.

3.19 Curvature

Definition 3.56 (Riemann curvature tensor)
The Riemann curvature tensor *is a map Fock*

$$R(-,-)- : \mathscr{V}(\mathscr{M}) \times \mathscr{V}(\mathscr{M}) \times \mathscr{V}(\mathscr{M}) \longrightarrow \mathscr{V}(\mathscr{M}).$$

such that for all triples of vector fields $\boldsymbol{V}$, $\boldsymbol{W}$, $\boldsymbol{Z}$, the vector field $R(\boldsymbol{V},\boldsymbol{W})\boldsymbol{Z}$ is defined as

$$R(\boldsymbol{V},\boldsymbol{W})\boldsymbol{Z} = D\left(\boldsymbol{V},\left(D\left(\boldsymbol{W},\boldsymbol{Z}\right)\right)\right) - D\left(\boldsymbol{W},\left(D\left(\boldsymbol{V},\boldsymbol{Z}\right)\right)\right) - D\left(\left[\boldsymbol{V},\boldsymbol{W}\right],\boldsymbol{Z}\right).$$

If we choose a basis on the tangent space $\{\boldsymbol{e}_i\}_{i=1,\dots,m}$ and let $\{\boldsymbol{E}^j\}_{j=1,\dots,m}$ be its dual basis the components of the Riemann tensor are defined as

$$R^l{}_{ijk} \overset{\text{def.}}{=} \boldsymbol{E}^l(R(\boldsymbol{e}_j,\boldsymbol{e}_k)\boldsymbol{e}_i).$$

If we define the second covariant derivative of a vector field $\boldsymbol{Z}$ as the covariant derivative of the covariant derivative of $\boldsymbol{Z}$, i.e. $D\left(,D\left(,Z\right)\right) = D\left(,\left(D\left(,Z\right)\right)\right)$, then in component notation we have

$$Z^l{}_{;ij} = Z^l{}_{;i;j}.$$

We will now compute explicitly $D\left(\boldsymbol{V},\left(D\left(\boldsymbol{W},\boldsymbol{Z}\right)\right)\right)$ in its local form, i.e. when a given basis $\{\boldsymbol{e}_i\}_{i=1,\dots,m}$ in the tangent space is fixed, so that $\boldsymbol{V} = \sum_i^{1,m} v_i\boldsymbol{e}_i$, $\boldsymbol{W} = \sum_j^{1,m} w_j\boldsymbol{e}_j$ and $\boldsymbol{Z} = \sum_k^{1,m} z_k\boldsymbol{e}_k$. We start with

$$\begin{aligned}
D\left(\boldsymbol{W},\boldsymbol{Z}\right) &= D\left(\boldsymbol{W},\boldsymbol{Z},\right) \\
&= D\left(\sum_j^{1,m} w_j\boldsymbol{e}_j, \sum_k^{1,m} z_k\boldsymbol{e}_k\right) \\
&= \sum_{j,k}^{1,m} D\left(w_j\boldsymbol{e}_j, z_k\boldsymbol{e}_k\right) \\
&= \sum_{j,k}^{1,m} D\left(\boldsymbol{e}_j, z_k\boldsymbol{e}_k\right) w_j
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,k}^{1,m} [\boldsymbol{e}_j(z_k) w_j \boldsymbol{e}_k + D(\boldsymbol{e}_j, \boldsymbol{e}_k) w_j z_k] \\
&= \sum_{j,k}^{1,m} \boldsymbol{e}_j(z_k) w_j \boldsymbol{e}_k + \sum_{h,j,k}^{1,m} \Gamma^h_{jk} w_j z_k \boldsymbol{e}_h \\
&= \sum_{j,h}^{1,m} w_j \boldsymbol{e}_j(z_h) \boldsymbol{e}_h + \sum_{h,j,k}^{1,m} \Gamma^h_{jk} w_j z_k \boldsymbol{e}_h \\
&= \sum_{h,j}^{1,m} w_j \left(\partial_j z_h + \sum_{k}^{1,m} \Gamma^h_{jk} z_k \right) \boldsymbol{e}_h \\
&= \sum_{h,j}^{1,m} z_{h;j} w_j \boldsymbol{e}_h,
\end{aligned}
$$

so that

$$
(D(\boldsymbol{W}, \boldsymbol{Z}))_i = \sum_{k}^{1,m} z_{i;k} w_k.
$$

Generalizing we then have

$$
\begin{aligned}
D(\boldsymbol{V}, (D(\boldsymbol{W}, \boldsymbol{Z}))) &= \sum_{h,j}^{1,m} (D(\boldsymbol{W}, \boldsymbol{Z}))_{h;j} v_j \boldsymbol{e}_h \\
&= \sum_{h,j}^{1,m} (\sum_{k}^{1,m} z_{h;k} w_k)_{;j} v_j \boldsymbol{e}_h \\
&= \sum_{h,j,k}^{1,m} (z_{h;k} w_k)_{;j} v_j \boldsymbol{e}_h,
\end{aligned}
$$

so that

$$
(D(\boldsymbol{V}, (D(\boldsymbol{W}, \boldsymbol{Z}))))_i = \sum_{j,k}^{1,m} (z_{i;k} w_k)_{;j} v_j.
$$

We remember that in terms of the dual basis $\{\boldsymbol{E}_k\}_{k=1,\ldots,m}$ we can also write

$$
(D(\boldsymbol{W}, \boldsymbol{Z}))_i = \boldsymbol{E}_i(D(\boldsymbol{W}, \boldsymbol{Z}))
$$

or, of course,

$$
(D(\boldsymbol{V}, (D(\boldsymbol{W}, \boldsymbol{Z}))))_i = \boldsymbol{E}_i(D(\boldsymbol{V}, (D(\boldsymbol{W}, \boldsymbol{Z})))).
$$

Proposition 3.38 (Riemann tensor and covariant derivatives)
The Riemann tensor expresses the non-commutativity of the second covariant derivatives of a vector field.

Proof:

We are going to use the definitions above. From the definition of the Riemann tensor we can extract the components thanks to

$$\begin{aligned}
\boldsymbol{E}_l(R(\boldsymbol{V},\boldsymbol{W})\boldsymbol{Z}) &= \boldsymbol{E}_l(D(\boldsymbol{V},(D(\boldsymbol{W},\boldsymbol{Z})))) - \boldsymbol{E}_l(D(\boldsymbol{W},(D(\boldsymbol{V},\boldsymbol{Z})))) + \\
&\quad -\boldsymbol{E}_l(D([\boldsymbol{V},\boldsymbol{W}],\boldsymbol{Z})) \\
&= \sum_{j,k}^{1,m}(z_{l;k}w_k)_{;j}v_j - \sum_{j,k}^{1,m}(z_{l;k}v_k)_{;j}w_j + \\
&\quad -\sum_{k}^{1,m} z_{l;k}[\boldsymbol{V},\boldsymbol{W}]_k \\
&= \sum_{j,k}^{1,m}(z_{l;k;j}w_k v_j + z_{l;k}w_{k;j}v_j) + \\
&\quad -\sum_{j,k}^{1,m}(z_{l;k;j}v_k w_j + z_{l;k}v_{k;j}w_j) + \\
&\quad -\sum_{j,k}^{1,m} z_{l;k}(w_{k;j}v_j - v_{k;j}w_j) \\
&= \sum_{j,k}^{1,m} z_{l;k;j}w_k v_j - \sum_{j,k}^{1,m} z_{l;k;j}v_k w_j \\
&= \sum_{j,k}^{1,m} z_{l;kj}w_k v_j - \sum_{j,k}^{1,m} z_{l;jk}v_j w_k \\
&= \sum_{j,k}^{1,m} (z_{l;kj} - z_{l;jk})\, v_j w_k.
\end{aligned}$$

Of course the left-hand side gives

$$\boldsymbol{E}_l(R(\boldsymbol{V},\boldsymbol{W})\boldsymbol{Z}) = \sum_{i,j,k}^{1,m} R^l{}_{ijk} v_j w_k z_i$$

so that

$$\sum_{j,k}^{1,m} (z_{l;kj} - z_{l;jk})\, v_j w_k = \sum_{j,k}^{1,m} \left(\sum_{i}^{1,m} R^l{}_{ijk} z_i \right) v_j w_k$$

and, since $\boldsymbol{V}$ and $\boldsymbol{W}$ are arbitrary vectors,

$$\sum_{i}^{1,m} R^l{}_{ijk} z_i = z_{l;kj} - z_{l;jk}.$$

□

Proposition 3.39 (Riemann tensor and coordinate basis)
In a coordinate basis, the Riemann tensor can be expressed in terms of the

connection as

$$R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \sum_a^{1,m} \left(\Gamma^i_{ka} \Gamma^a_{lj} - \Gamma^i_{la} \Gamma^a_{kj} \right).$$

Proof:

When we consider a 1-form $\boldsymbol{\omega}$ and three vector fields $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ the properties of the connection imply

$$D(\boldsymbol{X}, \boldsymbol{\eta} \otimes D(\boldsymbol{Y}, \boldsymbol{Z})) = D(\boldsymbol{X}, \boldsymbol{\eta}) \otimes D(\boldsymbol{Y}, \boldsymbol{Z}) + \boldsymbol{\eta} \otimes D(\boldsymbol{X}, D(\boldsymbol{Y}, \boldsymbol{Z})).$$

But the covariant derivative preserves contractions, so that the above implies:

$$\boldsymbol{\eta}(D(\boldsymbol{X}, D(\boldsymbol{Y}, \boldsymbol{Z}))) = \boldsymbol{X}(\boldsymbol{\eta}(D(\boldsymbol{Y}, \boldsymbol{Z}))) - (D(\boldsymbol{X}, \boldsymbol{\eta}))(D(\boldsymbol{Y}, \boldsymbol{Z})).$$

We can now consider the components of the Riemann tensor, i.e. once we fix a basis $\{\boldsymbol{e}_i\}_{i=1,\dots m}$ in the tangent space and the corresponding dual basis $\{\boldsymbol{E}_i\}_{i=1,\dots m}$ in the cotangent space, the $R^l{}_{ijk}$ defined above

$$R^i{}_{jkl} = \boldsymbol{E}_i(R(\boldsymbol{e}_k, \boldsymbol{e}_l)\boldsymbol{e}_j).$$

We now have

$$\begin{aligned}
\boldsymbol{E}_i(R(\boldsymbol{e}_k, \boldsymbol{e}_l)\boldsymbol{e}_j) &= \boldsymbol{E}_i(D(\boldsymbol{e}_k, D(\boldsymbol{e}_l, \boldsymbol{e}_j))) - \boldsymbol{E}_i(D(\boldsymbol{e}_l, D(\boldsymbol{e}_k, \boldsymbol{e}_j))) + \\
&\quad - \boldsymbol{E}_i(D([\boldsymbol{e}_k, \boldsymbol{e}_l], \boldsymbol{e}_j)) \\
&= \boldsymbol{e}_k(\boldsymbol{E}_i(D(\boldsymbol{e}_l, \boldsymbol{e}_j))) - (D(\boldsymbol{e}_k, \boldsymbol{E}_i))(D(\boldsymbol{e}_l, \boldsymbol{e}_j)) + \\
&\quad - \boldsymbol{e}_l(\boldsymbol{E}_i(D(\boldsymbol{e}_k, \boldsymbol{e}_j))) + (D(\boldsymbol{e}_l, \boldsymbol{E}_i))(D(\boldsymbol{e}_k, \boldsymbol{e}_j)) \\
&\quad - \boldsymbol{E}_i(D([\boldsymbol{e}_k, \boldsymbol{e}_l], \boldsymbol{e}_j)).
\end{aligned}$$

If we specialize to a coordinate basis in the tangent space and to its dual, the last term vanishes, because so so the Lie Brackets and the above turns into

$$\begin{aligned}
R^i{}_{jkl} &= \frac{\partial}{\partial x_k}(dx^i(\sum_a^{1,m} \Gamma^a_{lj} \frac{\partial}{\partial x_a})) - (-\sum_a^{1,m} \Gamma^i_{ka} dx^a)(\sum_b^{1,m} \Gamma^b_{lj} \frac{\partial}{\partial x_b}) + \\
&\quad - \frac{\partial}{\partial x_l}(dx^i(\sum_a^{1,m} \Gamma^a_{kj} \frac{\partial}{\partial x_a})) + (-\sum_a^{1,m} \Gamma^i_{la} dx^a)(\sum_b^{1,m} \Gamma^b_{kj} \frac{\partial}{\partial x_b}) \\
&= \frac{\partial}{\partial x_k}(\Gamma^i_{lj}) + \sum_a^{1,m} \Gamma^i_{ka} \Gamma^a_{lj} + \\
&\quad - \frac{\partial}{\partial x_l}(\Gamma^i_{kj}) - \sum_a^{1,m} \Gamma^i_{la} \Gamma^a_{kj} \\
&= \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \sum_a^{1,m} \left[\Gamma^i_{ka} \Gamma^a_{lj} - \Gamma^i_{la} \Gamma^a_{kj} \right]
\end{aligned}$$

as stated.

□

Proposition 3.40 (Properties of the Riemann tensor)
The Riemann tensor has the following symmetries:

$$\begin{aligned} R^i{}_{jab} &= -R^i{}_{jba} \\ R^i{}_{abc} + R^i{}_{bca} + R^i{}_{cab} &= 0. \end{aligned} \tag{3.23}$$

Moreover it satisfies the Bianchi Identities, i.e.

$$R^i{}_{jab;c} + R^i{}_{jbc;a} + R^i{}_{jca;b} = 0.$$

The above can also be shortly written as

$$\begin{aligned} R^i{}_{j(ab)} &= 0 \\ R^i{}_{[abc]} &= 0 \\ R^i{}_{j[ab;c]} &= 0 \end{aligned} \tag{3.24}$$

Definition 3.57 (Ricci tensor)
The Ricci R_{ij} tensor is the trace *of the Riemann tensor, i.e.*

$$R_{ij} = \sum_k^{1,m} R^k{}_{ikj}.$$

Till now we have assumed to have a generic connection. If a metric is defined on the manifold $\mathscr{M}$ and we consider the unique symmetric connection compatible with the metric, additional properties of the curvature follows. Moreover we can use the metric to raise and lower indices: thus we can define

$$R_{ijkl} \stackrel{\text{def.}}{=} g_{ia} R^a{}_{jkl}.$$

Proposition 3.41 (Additional symmetries of the Riemann tensor)
Let us consider the unique symmetric compatible connection derived by a metric. The Riemann tensor then satisfies the additional symmetries

$$\begin{aligned} R_{abij} &= -R_{baij} (\text{or equivalently} R_{(ab)ij} = 0) \\ R_{abij} &= R_{ijab}. \end{aligned}$$

Proposition 3.42 (Symmetries of the Ricci tensor)
Let us consider the unique symmetric compatible connection derived by a metric. The Ricci tensor is symmetric, i.e.

$$R_{ij} = R_{ji} (\text{or equivalently} R_{[ij]} = 0).$$

Definition 3.58 (Ricci scalar)
Let us consider the unique symmetric compatible connection derived by a metric. The Ricci scalar is the trace of the Ricci tensor, i.e.

$$R \stackrel{\text{def.}}{=} R^i{}_i = g^{ij} R_{ij}.$$

Definition 3.59 (Einstein tensor)
Let us consider the unique symmetric compatible connection derived by a metric. The Einstein tensor is the symmetric tensor defined as

$$G_{ij} \stackrel{\text{def.}}{=} R_{ij} - \frac{1}{2} g_{ij} R.$$

Proposition 3.43 (Differential identities of curvature tensors)
Let us consider the unique symmetric compatible connection derived by a metric. Then the following differential identities hold

$$R^a{}_{jkl;a} = R_{jl;k} - R_{jk;l}$$

$$R^a{}_{i;a} = \frac{1}{2} R_{;i}.$$

In particular the Einstein tensor is divergence-less, i.e.

$$G^{ij}{}_{;i} = 0.$$