Chapter 20

Lecture 20

20.1 Conservation of energy in classical mechanics

Let us consider a classical system with 1 degree of freedom, described by the generalized coordinate q. Let the system admit a Lagrangian formulation, and let $L\left(q, \frac{dq}{dt}, t\right)$ be the Lagrangian of the system. In terms of the Lagrangian the dynamics of the system is described by the Euler-Lagrange equations, i.e.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

We now make the additional hypothesis that the Lagrangian *does not explicitly* depend on the time t, i.e. mathematically that

$$\frac{\partial L}{\partial t} = 0.$$

In this case we have

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t} \\ &= \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right). \end{aligned} \tag{20.1}$$

Between the second and the third line we have used our hypothesis that the Lagrangian does not depend explicitly from the parameter t and in the last equality we have used that the equations of motion are satisfied. We thus get the equality

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right),$$

i.e.

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0.$$

Thus if the Lagrangian does not depend explicitly on time, the quantity

$$\dot{q}\frac{\partial L}{\partial \dot{q}} - L \tag{20.2}$$

is an *integral of the motion*[1], i.e. it is conserved by the dynamics of the system. We reformulate this fact in the following form: the independence of the Lagrangian from the parameter t implies the conservation of the quantity (20.2).

20.2 Conservation laws in a special relativistic field theory

Let us consider a theory consisting of N fields $\{\phi^{(i)}\}_{i=1,\dots,N}$, described by the Lagrangian density $\mathcal{L}(x^\mu, \phi^{(i)}, \partial_\nu \phi^{(j)})$. As we saw in lecture 3 the dynamics of the theory is described by the Euler-Lagrange equations,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})}\right) = \frac{\partial \mathcal{L}}{\partial \phi^{(i)}} \quad i = 1, \dots, N.$$

Let us now make the additional hypothesis that the Lagrangian does not depend explicitly from x^μ, i.e. that the dependence from x^μ always happens through the fields $\phi^{(i)}$ and their derivatives $\partial_\mu \phi^{(i)}$. In this case we have

$$\begin{aligned}
\partial_\nu \mathcal{L} &= \sum_i^{1,N} \frac{\partial \mathcal{L}}{\partial \phi^{(i)}} \partial_\nu \phi^{(i)} + \sum_i^{1,N} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})} \partial_\nu \partial_\mu \phi^{(i)} \\
&= \sum_i^{1,N} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})}\right) \partial_\nu \phi^{(i)} + \sum_i^{1,N} \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^{(i)})} \partial_\mu \partial_\nu \phi^{(i)} \\
&= \sum_i^{1,N} \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})}\right) (\partial_\nu \phi^{(i)}) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^{(i)})} \partial_\mu (\partial_\nu \phi^{(i)})\right] \\
&= \sum_i^{1,N} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})} (\partial_\nu \phi^{(i)})\right) \\
&= \partial_\mu \sum_i^{1,N} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})} (\partial_\nu \phi^{(i)})\right).
\end{aligned} \tag{20.3}$$

Again, we remember our hypothesis that the dependence of $\mathcal{L}$ from x^μ is only through the fields $\phi^{(i)}$ and their derivatives in the first line. We then use the

[1] Actually, if we remember that

$$p = \frac{\partial L}{\partial \dot{q}}$$

we see that the conserved quantity is just the Hamiltonian of the system,

$$H = p\dot{q} - L.$$

field equations in the second line. The final result is then

$$\delta^\mu_\nu \partial_\mu \mathcal{L} = \partial_\mu \sum_i^{1,N} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})} (\partial_\nu \phi^{(i)}) \right),$$

or, which is the same,

$$\partial_\mu [\delta^\mu_\nu \mathcal{L}] = \partial_\mu \sum_i^{1,N} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})} (\partial_\nu \phi^{(i)}) \right),$$

so that

$$\partial_\mu T^\mu{}_\nu = 0,$$

where we have defined

$$T^\mu{}_\nu = \sum_i^{1,N} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})} (\partial_\nu \phi^{(i)}) \right) - \delta^\mu_\nu \mathcal{L}.$$

Definition 20.1 (Stress Energy Tensor)
Let us consider a Field Theory consisting of N fields $\phi^{(i)}$ in n dimensions, that admits a Lagrangian formulation in terms of a Lagrangian density $\mathcal{L}$. The quantity

$$T^\mu{}_\nu = \sum_i^{1,N} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^{(i)})} (\partial_\nu \phi^{(i)}) \right) - \delta^\mu_\nu \mathcal{L}$$

is called the Stress-Energy tensor *of the fields.*

Proposition 20.1 (Local conservation laws)
If in the Lagrangian formulation of a field theory of N fields $\phi^{(i)}$ in n dimensions the Lagrangian density does not depend explicitly on the coordinates, then the stress-energy tensor is locally conserved,

$$\partial_\mu T^\mu{}_\nu = 0,$$

i.e. its divergence is zero.

20.3 Conservation laws and general covariance

Let us consider again a theory consisting of N fields $\{\phi^{(i)}\}_{i=1,\ldots,N}$, on a Lorentzian manifold $(\mathscr{M}, \mathcal{L}, \langle -, - \rangle)$. The system will be now described by a Lagrangian density $\mathcal{L}$ which is again a function of the fields, their first derivatives, the metric tensor and, eventually, the first derivatives of the metric tensor:

$$\mathcal{L} = \mathcal{L}(\phi^{(i)}, \partial_\nu \phi^{(j)}, g_{\alpha\beta}, \partial_\gamma g_{\alpha\beta}).$$

The action for our theory is then

$$\mathcal{S} = \int_{\mathscr{M}} d^4x \sqrt{-g} \mathcal{L}.$$

Let us now consider a change of coordinates of the form

$$\tilde{x}^\mu = x^\mu + \delta x^\mu \tag{20.4}$$

such that "δx^μ are small quantities"[2]. The metric tensor under this change of coordinates becomes $\tilde{g}_{\alpha\beta}(\tilde{x}^\mu)$ and can be expressed as

$$\begin{aligned}
\tilde{g}_{\alpha\beta}(\tilde{x}^\mu) &= g_{\rho\sigma}(x^\nu)\frac{\partial x^\rho}{\partial \tilde{x}^\alpha}\frac{\partial x^\sigma}{\partial \tilde{x}^\beta} \\
&= g_{\rho\sigma}(x^\nu)\left(\tilde{x}^\rho - \delta x^\rho\right)_{,\alpha}\left(\tilde{x}^\sigma - \delta x^\sigma\right)_{,\beta} \\
&= g_{\rho\sigma}(x^\nu)\left(\delta^\rho_\alpha - [\delta x^\rho]_{,\alpha}\right)\left(\delta^\sigma_\beta - [\delta x^\sigma]_{,\beta}\right) \\
&\approx g_{\alpha\beta}(x^\nu) - g_{\rho\beta}(x^\nu)[\delta x^\rho]_{,\alpha} - g_{\alpha\sigma}(x^\nu)[\delta x^\sigma]_{,\beta}. \qquad (20.5)
\end{aligned}$$

We now express the left-hand side of the above chain of equalities in such a way that the variables x^μ appear as arguments of the metric $\tilde{g}_{alpha\beta}$. This can be obtained by expanding $\tilde{g}_{alpha\beta}$ in powers of δx^μ, at first order, i.e.

$$\begin{aligned}
\tilde{g}_{\alpha\beta}(\tilde{x}^\mu) &= \tilde{g}_{\alpha\beta}(x^\mu + \delta x^\mu) \\
&\approx \tilde{g}_{\alpha\beta}(x^\mu) + \partial_\gamma g_{\alpha\beta}(x^\mu)[\delta x^\gamma]. \qquad (20.6)
\end{aligned}$$

We can now combine results (20.5) and (20.6) to get

$$\tilde{g}_{\alpha\beta}(x^\mu) + \partial_\gamma g_{\alpha\beta}(x^\mu)\delta x^\gamma \approx g_{\alpha\beta}(x^\nu) - g_{\rho\beta}(x^\nu)[\delta x^\rho]_{,\alpha} - g_{\alpha\sigma}(x^\nu)[\delta x^\sigma]_{,\beta}$$

or, which is the same[3],

$$\tilde{g}_{\alpha\beta} \approx g_{\alpha\beta} - \partial_\gamma g_{\alpha\beta}\delta x^\gamma - g_{\rho\beta}[\delta x^\rho]_{,\alpha} - g_{\alpha\sigma}[\delta x^\sigma]_{,\beta}.$$

Let us perform a further elaboration of the above result:

$$\begin{aligned}
\tilde{g}_{\alpha\beta} &\approx g_{\alpha\beta} - \partial_\gamma g_{\alpha\beta}[\delta x^\gamma] - g_{\rho\beta}[\delta x^\rho]_{,\alpha} - g_{\alpha\rho}[\delta x^\rho]_{,\beta} \\
&= g_{\alpha\beta} - \partial_\gamma g_{\alpha\beta}[\delta x^\gamma] - g_{\rho\beta}[\delta x^\rho]_{,\alpha} - g_{\alpha\rho}[\delta x^\rho]_{,\beta} + \\
&\quad + \Gamma^\rho_{\alpha\gamma}g_{\rho\beta}[\delta x^\gamma] + \Gamma^\rho_{\beta\gamma}g_{\alpha\rho}[\delta x^\gamma] - \Gamma^\rho_{\alpha\gamma}g_{\rho\beta}[\delta x^\gamma] - \Gamma^\rho_{\beta\gamma}g_{\alpha\rho}[\delta x^\gamma] \\
&= g_{\alpha\beta} - \left(\partial_\gamma g_{\alpha\beta} - \Gamma^\rho_{\alpha\gamma}g_{\rho\beta} - \Gamma^\rho_{\beta\gamma}g_{\alpha\rho}\right)[\delta x^\gamma] + \\
&\quad + g_{\rho\beta}[\delta x^\rho]_{,\alpha} - \Gamma^\rho_{\alpha\gamma}g_{\rho\beta}[\delta x^\gamma] + g_{\alpha\rho}[\delta x^\rho]_{,\beta} - \Gamma^\rho_{\beta\gamma}g_{\alpha\rho}[\delta x^\gamma] \\
&= g_{\alpha\beta} - g_{\alpha\beta;\gamma}[\delta x^\gamma] + \\
&\quad - g_{\rho\beta}\left([\delta x^\rho]_{,\alpha} + \Gamma^\rho_{\alpha\gamma}[\delta x^\gamma]\right) - g_{\alpha\rho}\left([\delta x^\rho]_{,\beta} + \Gamma^\rho_{\beta\gamma}[\delta x^\gamma]\right) \\
&= g_{\alpha\beta} - g_{\rho\beta}[\delta x^\rho]_{;\alpha} - g_{\alpha\rho}[\delta x^\rho]_{;\beta} \\
&= g_{\alpha\beta} - [\delta x_\beta]_{;\alpha} - [\delta x_\alpha]_{;\beta}.
\end{aligned}$$

If we indicate with $\delta g_{\alpha\beta}$ the variation of the metric we obtain

$$\delta g_{\alpha\beta} = -[\delta x_\alpha]_{;\beta} - [\delta x_\beta]_{;\alpha}$$

and

$$\delta g^{\alpha\beta} = [\delta x^\alpha]^{;\beta} + [\delta x^\beta]^{;\alpha}.$$

[2]Our apologies for this treatment which is not so precise in terms of the manifold concepts we have developed before.

[3]We drop the difference from x^μ, which is now common to all terms.

We return now to our action principle, which we rewrite here for convenience

$$\mathcal{S} = \int_{\mathscr{M}} d^4x\sqrt{-g}\mathcal{L};$$

we remember that if we consider a variation of the above actions with respect to the fields $\boldsymbol{\phi}$, as we already saw in lecture 3 we obtain the Euler-Lagrange equations for the fields $\boldsymbol{\phi}$. Indicating with $\delta_{\boldsymbol{\phi}}$ this variation we thus have

$$\frac{\delta_{\boldsymbol{\phi}} S[\boldsymbol{g},\boldsymbol{\phi}]}{\delta\boldsymbol{\phi}} = 0 \quad \Leftrightarrow \quad \frac{\partial\mathcal{S}}{\partial\phi^k} - \sum_{\mu}^{1,n} \partial_\mu \left(\frac{\partial\mathcal{S}}{\partial(\partial_\mu\phi^k)}\right) = 0 \quad k = 1,\ldots,N$$

i.e. fields configurations that make the action stationary satisfy Euler-Lagrange equations, our field equations.

We are now going to consider instead a different variation, which we are going to indicate with δ and is the variation of the action under a transformation of coordinates of the type (20.4). Since the fields $\boldsymbol{\phi} = \boldsymbol{\phi}(x^\mu)$ are functions of x^μ our variation induces a change $\delta\boldsymbol{\phi}$. Under the assumption that the equations for our fields are satisfied, we are nevertheless going to ignore this variation: it would contribute a term multiplied by the Euler-Lagrange, which, we stress that again, is going to vanish because we are assuming that the field equations are satisfied. Then there is a variation induced by the change of coordinates in the metric field (and its derivatives). This variation is the one we are interested in. With the assumed convention on the meaning of δ we thus have:

$$\begin{aligned}
\delta\mathcal{S}[\boldsymbol{g},\boldsymbol{\phi}] &= \int_{\mathscr{M}} d^4x \left[\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\lambda g^{\mu\nu})}\delta(\partial_\lambda g^{\mu\nu})\right] \\
&= d^4x \int_{\mathscr{M}} \left[\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\lambda g^{\mu\nu})}\partial_\lambda(\delta g^{\mu\nu})\right] \\
&= \int_{\mathscr{M}} d^4x \left[\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} - \partial_\lambda\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\lambda g^{\mu\nu})}\right)\right]\delta g^{\mu\nu} + \\
&\qquad + \int_{\mathscr{M}} d^4x \partial_\lambda\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\lambda g^{\mu\nu})}\delta g^{\mu\nu}\right) \\
&= \int_{\mathscr{M}} d^4x \left[\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} - \partial_\lambda\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\lambda g^{\mu\nu})}\right)\right]\delta g^{\mu\nu} + \\
&\qquad + \int_{\partial\mathscr{M}} d^3y \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\lambda g^{\mu\nu})}\delta g^{\mu\nu} \\
&= \int_{\mathscr{M}} d^4x \left[\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} - \partial_\lambda\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\lambda g^{\mu\nu})}\right)\right]\delta g^{\mu\nu}
\end{aligned}$$

Let us now set

$$\frac{1}{2}\sqrt{-g}T^{\mu\nu} = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} - \partial_\lambda\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_\lambda g^{\mu\nu})}\right), \tag{20.7}$$

which is clearly a symmetric tensor. Then we have

$$\delta\mathcal{S}[\boldsymbol{g},\boldsymbol{\phi}] = \frac{1}{2}\int_{\mathscr{M}} d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathscr{M}} d^4x\sqrt{-g}T_{\mu\nu}([\delta x^\mu]^{;\nu} + [\delta x^\nu]^{;\mu}) \\
&= \int_{\mathscr{M}} d^4x\sqrt{-g}T_{\mu\nu}[\delta x^\mu]^{;\nu} \\
&= \int_{\mathscr{M}} d^4x\sqrt{-g}\left\{(T_{\mu\nu}[\delta x^\mu])^{;\nu} - T_{\mu\nu}{}^{;\nu}[\delta x^\mu]\right\} \\
&= \int_{\mathscr{M}} d^4x\sqrt{-g}\left\{(T^{\mu\nu}[\delta x_\mu])_{;\nu} - T^{\mu\nu}{}_{;\nu}[\delta x_\mu]\right\} \\
&= \int_{\mathscr{M}} d^4x\sqrt{-g}\frac{1}{\sqrt{-g}}\left(\sqrt{-g}T^{\mu\nu}[\delta x_\mu]\right)_{,\nu} - \int_{\mathscr{M}} d^4x\sqrt{-g}T^{\mu\nu}{}_{;\nu}[\delta x_\mu] \\
&= \int_{\mathscr{M}} d^4x\left(\sqrt{-g}T^{\mu\nu}[\delta x_\mu]\right)_{,\nu} - \int_{\mathscr{M}} d^4x\sqrt{-g}T^{\mu\nu}{}_{;\nu}[\delta x_\mu] \\
&= \int_{\partial\mathscr{M}} d^3y\left(\sqrt{-g}T^{\mu\nu}[\delta x_\mu]\right)_{,\nu} - \int_{\mathscr{M}} d^4x\sqrt{-g}T^{\mu\nu}{}_{;\nu}[\delta x_\mu] \\
&= -\int_{\mathscr{M}} d^4x\sqrt{-g}T^{\mu\nu}{}_{;\nu}[\delta x_\mu]. \qquad (20.8)
\end{aligned}
$$

From the arbitrariness of the $[\delta x_\mu]$ we obtain that the stationarity of the action with respect to the variation defined by δ implies

$$T^{\mu\nu}{}_{;\nu} = 0.$$

From the analogy of this equation with the one in definition 20.1 we are going to call $T^{\mu\nu}$ the energy momentum tensor and the equation above its conservation law. The stress energy tensor has the following physical interpretation:

T^{00}: energy density;

T^{i0}: momentum density;

T^{ij}: stress tensor (pressure tensor);

T^{00}: energy current density (energy density flow).

In the special relativistic case the stress-energy tensor does not appear explicitly symmetric as in the case of general covariance. It can nevertheless easily seen that the definition of the stress-energy tensor in this case is not unique. In particular, without invalidating the local conservation law, it is always possible to add to the stress-energy tensor of a theory with Lorentz invariance a tensor of the form

$$\partial_\alpha \Delta^{\alpha\mu\nu},$$

where

$$\Delta^{\alpha\mu\nu} = -\Delta^{\alpha\nu\mu};$$

exploiting this arbitrariness we can put always the stress-energy tensor in a symmetric form, without changing the momentum density and without affecting the conservation law. Moreover, the symmetry of the stress-energy tensor in a Lorentz invariant theory is equivalent to the conservation of the angular momentum of the fields.

20.4 Einstein equations

The stress energy tensor defined above acts as a source for the full set of Einstein equations, when matter content is present. In particular let us consider the action

$$\mathcal{S}[\boldsymbol{g}, \boldsymbol{\phi}] = \mathcal{S}_{\mathrm{G}}[\boldsymbol{g}] + \mathcal{S}_{\mathrm{M}}[\boldsymbol{g}, \boldsymbol{\phi}]$$

If we perform a variation of the above action, reproducing the computations we already did in the previous lectures, we obtain that the stationary action principle gives

$$G_{\mu\nu} = \kappa T^{\mu\nu},$$

where κ is a suitable constant that can be determined by requiring the correct Newtonian limit. Explicitly the above equations can be also written as

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R = \kappa T^{\mu}_{\nu}$$

or, which is the same,

$$R^{\mu}_{\nu} = \kappa \left(T^{\mu}_{\nu} - \delta^{\mu}_{\nu} T \right),$$

where T is the trace of the stress-energy tensor. The above set of equations is the set of *Einstein field equations*. Note that when there is a matter content (matter fields) the full system of differential equations is the coupled system of Einstein field equations and of the Euler-Lagrange equations for the fields. We also quote that the set of field equations for the other (i.e. non-gravitational) fields, is implied by the conservation law ${T^{\mu\nu}}_{;\nu} = 0$ or, which is the same, by ${G^{\mu\nu}}_{;\nu} = 0$.