Chapter 15

Lecture 15

15.1 Riemannian (Lorentzian) geometry - 2 -

15.1.1 Interplay between connection and metric

Definition 15.1 (Compatibility condition)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold with a metric $\langle -, - \rangle$. A connection $D(-,-)$ is compatible *with the metric $\langle -, - \rangle$ if $\forall \boldsymbol{V}\ \boldsymbol{W}$, parallel vector fields along an arbitrary given curve σ, it holds that $\langle \boldsymbol{V}, \boldsymbol{W} \rangle$ is constant along σ.*

This means that $\forall t$ for which σ is defined, the parallel translation along σ from $\sigma(0)$ to $\sigma(t)$ defines an isometry between $\mathscr{M}_{\sigma(0)}$ and $\mathscr{M}_{\sigma(t)}$.

Proposition 15.1 (I characterization of compatible connections)
A connection $D(-,-)$ on a manifold $(\mathscr{M}, \mathcal{F})$ with metric $\langle -, - \rangle$ is compatible with the metric if and only if $\forall \boldsymbol{V}$, $\boldsymbol{W}$, parallel vector fields along an arbitrary curve σ, the equality

$$\frac{d}{dt} \langle \boldsymbol{V}(t), \boldsymbol{W}(t) \rangle = \left\langle \frac{D\boldsymbol{V}}{dt}, \boldsymbol{W} \right\rangle + \left\langle \frac{D\boldsymbol{W}}{dt}, \boldsymbol{V} \right\rangle$$

is identically satisfied.

Proof:

$\Rightarrow$) Let us choose $\boldsymbol{P}_{(1)}$, ..., $\boldsymbol{P}_{(m)}$, m vector fields along σ which are orthonormal at a given point of σ. We can assume without restriction that they are parallel along σ (since given a vector at a point of a curve, to parallel propagate it along the curve we have only to solve the differential equations (10.5) = 0 with exactly the components of this vector as initial conditions). Then they are also orthonormal along σ, since their orthonormalization condition

$$\langle \boldsymbol{P}_{(i)}(t), \boldsymbol{P}_{(j)}(t) \rangle = \delta_{ij}$$

is preserved along σ precisely because the $\boldsymbol{P}_{(i)}$ are parallel along σ.

At every point of σ, we can thus write two arbitrary vector fields $\boldsymbol{V}$, $\boldsymbol{W}$ in terms of the orthonormal basis composed by the m vectors $\boldsymbol{P}_{(i)}$, i.e.

$$\begin{aligned}\boldsymbol{V}(t) &= \sum_i^{1,m} v^i(t)\boldsymbol{P}_{(i)}(t)\\ \boldsymbol{W}(t) &= \sum_i^{1,m} w^i(t)\boldsymbol{P}_{(i)}(t).\end{aligned}$$

Moreover, since $\boldsymbol{P}_{(i)}$, $i = 1, \ldots, m$, are parallel vector fields, we also have

$$\begin{aligned}\frac{D\boldsymbol{V}(t)}{dt} &= \sum_i^{1,m} \frac{dv^i(t)}{dt}\boldsymbol{P}_{(i)}(t) + \sum_i^{1,m} v^i(t)\frac{D\boldsymbol{P}_{(i)}(t)}{dt}\\ &= \sum_i^{1,m} \frac{dv^i(t)}{dt}\boldsymbol{P}_{(i)}(t)\\ \frac{D\boldsymbol{W}(t)}{dt} &= \sum_j^{1,m} \frac{dw^j(t)}{dt}\boldsymbol{P}_{(j)}(t) + \sum_j^{1,m} w^j(t)\frac{D\boldsymbol{P}_{(j)}(t)}{dt}\\ &= \sum_j^{1,m} \frac{dw^j(t)}{dt}\boldsymbol{P}_{(j)}(t).\end{aligned}$$

Remembering that the $\boldsymbol{P}_i(t)$ are orthonormal along σ, we can now compute

$$\begin{aligned}\left\langle \frac{D\boldsymbol{V}}{dt}, \boldsymbol{W}\right\rangle &= \left\langle \sum_i^{1,m} \frac{dv^i(t)}{dt}\boldsymbol{P}_{(i)}(t), \sum_j^{1,m} w^j(t)\boldsymbol{P}_{(j)}(t)\right\rangle\\ &= \sum_{i,j}^{1,m} \frac{dv^i(t)}{dt} w^j(t)\left\langle \boldsymbol{P}_{(i)}(t), \boldsymbol{P}_{(j)}(t)\right\rangle\\ &= \sum_{i,j}^{1,m} \frac{dv^i(t)}{dt} w^j(t)\delta_{ij}\\ &= \sum_i^{1,m} \frac{dv^i(t)}{dt} w_i(t)\end{aligned}$$

and exchanging $\boldsymbol{V}$ with $\boldsymbol{W}$

$$\left\langle \boldsymbol{V}, \frac{D\boldsymbol{W}}{dt}\right\rangle = \sum_i^{1,m} \frac{dw^i(t)}{dt} v_i(t);$$

Summing the last two result we thus get

$$\begin{aligned}\left\langle \frac{D\boldsymbol{V}}{dt}, \boldsymbol{W}\right\rangle + \left\langle \boldsymbol{V}, \frac{D\boldsymbol{W}}{dt}\right\rangle &= \sum_i^{1,m} \left(\frac{dv^i(t)}{dt} w_i(t) + \frac{dw^i(t)}{dt} v_i(t)\right)\\ &= \frac{d}{dt}\left(\sum_i^{1,m} v^i w_i\right)\\ &= \frac{d}{dt}\left\langle \boldsymbol{V}, \boldsymbol{W}\right\rangle,\end{aligned}$$

which completes the proof of this implication.

$\Leftarrow$) If $\boldsymbol{V}$ and $\boldsymbol{W}$ are parallel along σ then $D\boldsymbol{V}/dt = D\boldsymbol{W}/dt = 0$, i.e.

$$\frac{d}{dt}\left\langle \boldsymbol{V}(t), \boldsymbol{W}(t)\right\rangle = 0$$

so that

$$\langle \boldsymbol{V}(t), \boldsymbol{W}(t) \rangle = \text{const.}$$

and D is compatible with the metric.

□

Proposition 15.2 (II characterization of compatible connections) *A connection $D(-,-)$ on a manifold $(\mathscr{M}, \mathcal{F})$ with metric $\langle -,- \rangle$ is compatible with the metric if and only if $\forall \boldsymbol{V}$, $\boldsymbol{W}$, $\boldsymbol{Z}$ vector fields on $\mathscr{M}$ it holds that*

$$\boldsymbol{Z}(\langle \boldsymbol{V}, \boldsymbol{W} \rangle) = \langle D(\boldsymbol{Z}, \boldsymbol{V}), \boldsymbol{W} \rangle + \langle D(\boldsymbol{Z}, \boldsymbol{W}), \boldsymbol{V} \rangle . \tag{15.1}$$

Proof:

Let σ be a differentiable curve on $\mathscr{M}$ such that

$$\begin{aligned} \sigma(0) &= \mathtt{m} \in \mathscr{M} \\ \dot{\boldsymbol{\sigma}}(0) &= \boldsymbol{Z}_{\mathtt{m}} \in \mathscr{M}_{\mathtt{m}}. \end{aligned}$$

Remembering these settings we preliminarily define the following quantities:

$$\boldsymbol{Z}_{\mathtt{m}}(\langle \boldsymbol{V}, \boldsymbol{W} \rangle) \tag{15.2}$$

$$\left.\frac{d}{dt}\right]_{t=0} \left\langle \boldsymbol{V}_{\sigma(t)}, \boldsymbol{W}_{\sigma(t)} \right\rangle \tag{15.3}$$

$$\left\langle \left.\frac{D\boldsymbol{V}}{dt}\right]_{t=0}, \boldsymbol{W} \right\rangle + \left\langle \left.\frac{D\boldsymbol{W}}{dt}\right]_{t=0}, \boldsymbol{V} \right\rangle \tag{15.4}$$

$$\langle D(\boldsymbol{Z}_{\mathtt{m}}, \boldsymbol{V}), \boldsymbol{W} \rangle + \langle D(\boldsymbol{Z}_{\mathtt{m}}, \boldsymbol{W}), \boldsymbol{V} \rangle \tag{15.5}$$

$\Rightarrow$) We now start with the direct implication. We can compute the directional derivative of the function $\langle \boldsymbol{U}, \boldsymbol{V} \rangle$ in the direction of $\boldsymbol{Z}_{\mathtt{m}}$ as the derivative along the curve σ at $t = 0$: since this directional derivative is a local expression it does not depend on the chosen curve, provided it has tangent vector $\boldsymbol{Z}_{\mathtt{m}}$ at $\mathtt{m}$. This says (15.2) = (15.3). Using proposition 15.1 we know that (15.3) = (15.4) and by definition of the covariant derivative along a curve (15.4) = (15.5). Thus (15.2) = (15.5) $\forall \mathtt{m} \in \mathscr{M}$, which is equivalent to (15.1), the result to be established.
$\Leftarrow$) To prove the converse we observe that now (15.1) holds, so that (15.2) = (15.5) $\forall \mathtt{m} \in \mathscr{M}$. But again, by the same considerations we made above, (15.2) = (15.3) and (15.5) = (15.4). So we have established that under the assumed conditions (15.3) = (15.4) i.e. that $\forall \boldsymbol{V}, \boldsymbol{W}$ vector fields along a curve σ

$$\frac{d}{dt} \langle \boldsymbol{V}, \boldsymbol{W} \rangle = \left\langle \frac{D\boldsymbol{V}}{dt}, \boldsymbol{W} \right\rangle + \left\langle \frac{D\boldsymbol{W}}{dt}, \boldsymbol{V} \right\rangle$$

so that proposition 15.1 assures the connection is compatible.

□

Proposition 15.3 (∃! symmetric compatible connection)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold and $\langle -, - \rangle$ a metric on $\mathscr{M}$. There exists one and only one symmetric connection on $\mathscr{M}$ compatible with the given metric.

Proof:

We will prove the uniqueness: let $(U, \phi) \in \mathcal{F}$ be a coordinate system with coordinate functions $x_1, \ldots, x_m$. As usual, we have that locally the connection can be expressed as

$$D(\partial x_i, \partial x_j) = \sum_{k}^{1,m} \Gamma^k_{ij} \partial x_k$$

and the Riemannian metric as

$$g_{mn} = \langle \partial x_m, \partial x_n \rangle .$$

The compatibility condition implies

$$\partial x_j \langle \partial x_k, \partial x_l \rangle = \langle D(\partial x_j, \partial x_k), \partial x_l \rangle + + \langle D(\partial x_j, \partial x_l), \partial x_k \rangle \qquad (15.6)$$

and permuting the indices j, k and l we also get

$$\begin{aligned} \partial x_k \langle \partial x_l, \partial x_j \rangle &= \langle D(\partial x_k, \partial x_l), \partial x_j \rangle + \qquad (15.7) \\ &\quad + \langle D(\partial x_k, \partial x_j), \partial x_l \rangle \\ \partial x_l \langle \partial x_j, \partial x_k \rangle &= \langle D(\partial x_l, \partial x_j), \partial x_k \rangle + \\ &\quad + \langle D(\partial x_l, \partial x_k), \partial x_j \rangle \\ &= \langle D(\partial x_j, \partial x_l), \partial x_k \rangle + \\ &\quad + \langle D(\partial x_k, \partial x_l), \partial x_j \rangle , \qquad (15.8) \end{aligned}$$

where in the last equality we have used the fact that the connection is symmetric and the arguments are element of a coordinate basis, so that result *2.* of proposition 10.1 applies. Summing side by side (15.6) and (15.7) and subtracting (15.8) we get

$$\begin{aligned} &\partial x_j \langle \partial x_k, \partial x_l \rangle + \partial x_k \langle \partial x_l, \partial x_j \rangle - \partial x_l \langle \partial x_j, \partial x_k \rangle = \\ &\quad = \langle D(\partial x_j, \partial x_k), \partial x_l \rangle + \langle D(\partial x_j, \partial x_l), \partial x_k \rangle + \\ &\quad\quad + \langle D(\partial x_k, \partial x_l), \partial x_j \rangle + \langle D(\partial x_k, \partial x_j), \partial x_l \rangle + \\ &\quad\quad - \langle D(\partial x_j, \partial x_l), \partial x_k \rangle - \langle D(\partial x_k, \partial x_l), \partial x_j \rangle \\ &\quad = \langle D(\partial x_j, \partial x_k), \partial x_l \rangle + \langle D(\partial x_k, \partial x_j), \partial x_l \rangle \\ &\quad = 2 \langle D(\partial x_j, \partial x_k), \partial x_l \rangle , \end{aligned}$$

where in the last line we again used the symmetry property of the connection.
The equality coming from the first and last lines can be rewritten, using the metric the and connection symbols in the chosen coordinate system that we have written at the beginning of this proof, as

$$-\partial_l g_{jk} + \partial_j g_{kl} + \partial_k g_{lj} = 2 \sum_{h}^{1,m} \Gamma^h_{jk} g_{hl}$$

or, acting with the inverse of the metric[1], as

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l}^{1,m}(g^{-1})_{il}\left(-\partial_l g_{jk} + \partial_j g_{kl} + \partial_k g_{lj}\right) \tag{15.9}$$

We leave the proof of the existence as an exercise. This is a routine procedure, where, taking (15.9) above, we show that the connection defined in term of exactly these connection symbols satisfies all the required properties.

□

Notation 15.1 (Compatible Symmetric Covariant Derivative)
When we consider the unique symmetric connection compatible with a metric on a manifold, we are going to use the following notation:

$$\boldsymbol{\nabla}_{\boldsymbol{V}}\boldsymbol{W} = D\left(\boldsymbol{V}, \boldsymbol{W}\right).$$

15.2 Curvature - 2 -

15.2.1 Curvature on Riemannian (Lorentzian) Manifolds

We already discussed the curvature tensor, i.e. the Riemann tensor, in Lectures 12 and 13. The properties we proved are valid for a generic connection. On the other hand we just saw that on a manifold where we can perform the scalar product of vectors, there is a privileged connection, namely the *only one* which is compatible with the metric is defined on $\mathscr{M}$. We also know that when a metric is present there is a natural isomorphism between tangent vectors and covectors, which is nothing but the operation of lowering/raising an index by the metric or ints inverse. Thus when a metric is present we can define

$$R_{ijkl} \stackrel{\text{def.}}{=} \sum_{a}^{1,m} g_{ia} R^a{}_{jkl}.$$

Proposition 15.4 (More symmetries of the Riemann tensor)
Let us consider the unique symmetric compatible connection derived by a metric. The Riemann tensor then satisfies the additional symmetries

$$\begin{aligned} R_{abij} &= -R_{baij} (\text{or equivalently } R_{(ab)ij} = 0) \\ R_{abij} &= R_{ijab}. \end{aligned}$$

[1]Here we will denote the inverse metric tensor as $(g^{-1})_{ik}$, i.e.

$$\sum_{k}^{1,m}(g^{-1})_{ik}g_{kj} = \sum_{k}^{1,m} g_{ik}(g^{-1})_{kj} = \delta_{ij}.$$

Of course we have $g_{ij} = g_{ji}$ and $(g^{-1})_{ij} = (g^{-1})_{ji}$.

Proposition 15.5 (Symmetries of the Ricci tensor)
Let us consider the unique symmetric compatible connection derived by a metric. The Ricci tensor is symmetric, i.e.

$$R_{ij} = R_{ji} (\text{or equivalently} R_{[ij]} = 0).$$

Definition 15.2 (Ricci scalar)
Let us consider the unique symmetric compatible connection derived by a metric. The Ricci scalar is the trace of the Ricci tensor, i.e.

$$R \stackrel{\text{def.}}{=} R^i{}_i = \sum_{i,j}^{1,m} g^{ij} R_{ij}.$$

Definition 15.3 (Einstein tensor)
Let us consider the unique symmetric compatible connection derived by a metric. The Einstein tensor is the symmetric tensor defined as

$$G_{ij} \stackrel{\text{def.}}{=} R_{ij} - \frac{1}{2} g_{ij} R.$$

Proposition 15.6 (Differential identities of curvature tensors)
Let us consider the unique symmetric compatible connection derived by a metric. Then the following differential identities hold

$$\begin{aligned} \sum_{a}^{1,m} R^a{}_{jkl;a} &= R_{jl;k} - R_{jk;l} \\ \sum_{a}^{1,m} R^a{}_{i;a} &= \frac{1}{2} R_{;i}. \end{aligned}$$

In particular the Einstein tensor is divergence-less, i.e.

$$\sum_{i}^{1,m} G^{ij}{}_{;i} = 0.$$