Chapter 13

Lecture 13

13.1 Curvature - 2 -

13.1.1 Components of the Riemann tensor, symmetries and Ricci tensor

Proposition 13.1 (Riemann tensor and coordinate basis)
In a coordinate basis, the Riemann tensor can be expressed in terms of the connection as

$$R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \sum_a^{1,m} \left(\Gamma^i_{ka} \Gamma^a_{lj} - \Gamma^i_{la} \Gamma^a_{kj} \right). \tag{13.1}$$

Proof:

When we consider a 1-form $\boldsymbol{\eta}$ and three vector fields $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ the properties of the connection imply

$$D(\boldsymbol{X}, \boldsymbol{\eta} \otimes D(\boldsymbol{Y}, \boldsymbol{Z})) = D(\boldsymbol{X}, \boldsymbol{\eta}) \otimes D(\boldsymbol{Y}, \boldsymbol{Z}) + \boldsymbol{\eta} \otimes D(\boldsymbol{X}, D(\boldsymbol{Y}, \boldsymbol{Z}))\,.$$

We now remember that one of the properties of the covariant derivative is that it preserve contractions. We will shortly apply this property to the relation above, but first we make the following observations.

1. $\boldsymbol{\eta}$ is a 1-form and $D(\boldsymbol{Y}, \boldsymbol{Z})$ is a vector field; thus the contraction of $\boldsymbol{\eta} \otimes D(\boldsymbol{Y}, \boldsymbol{Z})$ is the function
$$\boldsymbol{\eta}(D(\boldsymbol{Y}, \boldsymbol{Z}))\,;$$
When we consider $D(\boldsymbol{X}, \boldsymbol{\eta}(D(\boldsymbol{Y}, \boldsymbol{Z})))$ by the properties of the covariant derivative of a tensor we get
$$D(\boldsymbol{X}, \boldsymbol{\eta}(D(\boldsymbol{Y}, \boldsymbol{Z}))) = d(\boldsymbol{\eta}(D(\boldsymbol{Y}, \boldsymbol{Z})))(\boldsymbol{X}) = X(\boldsymbol{\eta}(D(\boldsymbol{Y}, \boldsymbol{Z}))),$$
where we also have used the definition of differential.
2. $D(\boldsymbol{X}, \boldsymbol{\eta})$ is a $(0,1)$-tensor field and $D(\boldsymbol{Y}, \boldsymbol{Z})$ is a vector field. Their contraction is the function obtained when the $(0,1)$-tensor field acts on the vector field, which we write
$$(D(\boldsymbol{X}, \boldsymbol{\eta}))(D(\boldsymbol{Y}, \boldsymbol{Z})).$$

3. $\boldsymbol{\eta}$ is a 1-form and $D(\boldsymbol{X}, D(\boldsymbol{Y}, \boldsymbol{Z}))$ is a vector field; again the contraction of $\boldsymbol{\eta} \otimes D(\boldsymbol{X}, D(\boldsymbol{Y}, \boldsymbol{Z}))$ is the function obtained by the action of the 1-form on the vector field, i.e.
$$\boldsymbol{\eta}(D(\boldsymbol{X}, D(\boldsymbol{Y}, \boldsymbol{Z}))).$$

We can rearrange the terms in the first equation of this proof as follows,

$$\boldsymbol{\eta} \otimes D(\boldsymbol{X}, D(\boldsymbol{Y}, \boldsymbol{Z})) = D(\boldsymbol{X}, \boldsymbol{\eta}) \otimes D(\boldsymbol{Y}, \boldsymbol{Z}) - D(\boldsymbol{X}, \boldsymbol{\eta} \otimes D(\boldsymbol{Y}, \boldsymbol{Z})),$$

and use the results in 3., 2., 1. above, together with the already recalled property that the covariant derivative preserves contractions, so that we obtain:

$$\boldsymbol{\eta}(D(\boldsymbol{X}, D(\boldsymbol{Y}, \boldsymbol{Z}))) = (D(\boldsymbol{X}, \boldsymbol{\eta}))(D(\boldsymbol{Y}, \boldsymbol{Z})) - \boldsymbol{X}(\boldsymbol{\eta}(D(\boldsymbol{Y}, \boldsymbol{Z}))). \tag{13.2}$$

We can now consider the components of the Riemann tensor, i.e. our sought $R^l{}_{ijk}$, which by definition can be written as

$$R^i{}_{jkl} = \boldsymbol{E}^i(R(\boldsymbol{e}_k, \boldsymbol{e}_l)\boldsymbol{e}_j),$$

after we have fixed a basis $\{\boldsymbol{e}_i\}_{i=1,\dots m}$ in the tangent space and the corresponding dual basis $\{\boldsymbol{E}^i\}_{i=1,\dots m}$ in the cotangent space. Using the definition of the Riemann tensor and then applying the preliminary result (13.2), we then have

$$\begin{aligned}
\boldsymbol{E}^i(R(\boldsymbol{e}_k, \boldsymbol{e}_l)\boldsymbol{e}_j) &= \boldsymbol{E}^i(D(\boldsymbol{e}_k, D(\boldsymbol{e}_l, \boldsymbol{e}_j))) - \boldsymbol{E}^i(D(\boldsymbol{e}_l, D(\boldsymbol{e}_k, \boldsymbol{e}_j))) + \\
&\quad -\boldsymbol{E}^i(D([\boldsymbol{e}_k, \boldsymbol{e}_l], \boldsymbol{e}_j)) \\
&= \boldsymbol{e}_k(\boldsymbol{E}^i(D(\boldsymbol{e}_l, \boldsymbol{e}_j))) - (D\left(\boldsymbol{e}_k, \boldsymbol{E}^i\right))(D(\boldsymbol{e}_l, \boldsymbol{e}_j)) + \\
&\quad -\boldsymbol{e}_l(\boldsymbol{E}^i(D(\boldsymbol{e}_k, \boldsymbol{e}_j))) + (D\left(\boldsymbol{e}_l, \boldsymbol{E}^i\right))(D(\boldsymbol{e}_k, \boldsymbol{e}_j)) \\
&\quad -\boldsymbol{E}^i(D([\boldsymbol{e}_k, \boldsymbol{e}_l], \boldsymbol{e}_j)).
\end{aligned}$$

If we specialize to a coordinate basis $\{\partial_a\}_{a=1,\dots,n}$ in the tangent space and to its dual $\{dx^a\}_{a=1,\dots,n}$, the last term vanishes, because so do the Lie Brackets, and the above turns into

$$\begin{aligned}
R^i{}_{jkl} &= \partial x_k(dx^i(\sum_a^{1,m} \Gamma^a_{lj}\partial x_a)) - (-\sum_a^{1,m} \Gamma^i_{ka} dx^a)(\sum_b^{1,m} \Gamma^b_{lj}\partial x_b) + \\
&\quad -\partial x_l(dx^i(\sum_a^{1,m} \Gamma^a_{kj}\partial x_a)) + (-\sum_a^{1,m} \Gamma^i_{la} dx^a)(\sum_b^{1,m} \Gamma^b_{kj}\partial x_b) \\
&= \partial x_k(\Gamma^i_{lj}) + \sum_a^{1,m} \Gamma^i_{ka}\Gamma^a_{lj} + \\
&\quad -\partial x_l(\Gamma^i_{kj}) - \sum_a^{1,m} \Gamma^i_{la}\Gamma^a_{kj} \\
&= \partial_k\Gamma^i_{lj} - \partial_l\Gamma^i_{kj} + \sum_a^{1,m} \left[\Gamma^i_{ka}\Gamma^a_{lj} - \Gamma^i_{la}\Gamma^a_{kj}\right]
\end{aligned}$$

as stated.

□

Proposition 13.2 (Properties of the Riemann tensor)
The Riemann tensor has the following symmetries:

$$
\begin{aligned}
R^i{}_{jab} &= -R^i{}_{jba} \\
R^i{}_{abc} + R^i{}_{bca} + R^i{}_{cab} &= 0.
\end{aligned} \tag{13.3}
$$

Moreover it satisfies the Bianchi identities, i.e.

$$
R^i{}_{jab;c} + R^i{}_{jbc;a} + R^i{}_{jca;b} = 0.
$$

The above can also be shortly written as

$$
\begin{aligned}
R^i{}_{j(ab)} &= 0 \\
R^i{}_{[abc]} &= 0 \\
R^i{}_{j[ab;c]} &= 0
\end{aligned} \tag{13.4}
$$

Definition 13.1 (Ricci tensor)
The Ricci R_{ij} tensor is the $1-2$ contraction *of the Riemann tensor, i.e.*

$$
R_{ij} = \sum_{k}^{1,m} R^k{}_{ikj}.
$$