Chapter 10

Lecture 10

10.1 Connections on manifolds - 2 -

10.1.1 Characterization of symmetric connections

Proposition 10.1 (Characterization of symmetric connections)
Let $D(-,-)$ be a connection on a manifold $\mathscr{M}, \mathcal{F}$ and $(U,\phi) \in \mathcal{F}$ a chart of $\mathscr{M}$ with coordinate functions $(x^1, \ldots, x^m)$. The following conditions are equivalent:

1. $D(-,-)$ *is symmetric;*
2. $D\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = D\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)$*;*
3. $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Proof:

$1 \Rightarrow 2$ Let us consider a symmetric connection. In a coordinate basis of $\mathscr{M}_{\mathtt{m}}$, as is the one induced by the given chart, the Lie Brackets of two arbitrary basis vectors vanish, i.e.

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

Thus

$$D\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) - D\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = 0$$

or

$$D\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = D\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right).$$

□

$2 \Rightarrow 3$ If we express

$$D\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = D\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right)$$

in terms of the connection symbols, the above equality becomes

$$\sum_{k}^{1,m} \left(\Gamma^k_{ij} - \Gamma^k_{ji} \right) \frac{\partial}{\partial x^k} = 0.$$

But, since $\{\partial/\partial x^k\}_{k=1,\ldots,m}$ is a basis of $\mathscr{M}_{\mathtt{m}}$ at each point $\mathtt{m} \in U \subset \mathscr{M}$, the $\partial/\partial x^k$ are linearly independent, i.e.

$$\Gamma^k_{ij} - \Gamma^k_{ji} = 0 \quad \Rightarrow \quad \Gamma^k_{ij} = \Gamma^k_{ji}.$$

$\square$

3 ⇒ 1 We consider to arbitrary vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$ and write them in a coordinate basis associated to a given chart (U, ϕ) with coordinate functions $(x_1, \ldots, x_m)$:

$$\begin{aligned} \boldsymbol{V} &= \sum_{i}^{1,m} v_i \frac{\partial}{\partial x_i} \\ \boldsymbol{W} &= \sum_{j}^{1,m} w_j \frac{\partial}{\partial x_j} \end{aligned} \tag{10.1}$$

We first compute

$$\begin{aligned} D(\boldsymbol{V}, \boldsymbol{W}) &= D\left(\sum_{i}^{1,m} v^i \frac{\partial}{\partial x^i}, \sum_{j}^{1,m} w^j \frac{\partial}{\partial x^j} \right) \\ &= \sum_{i,j}^{1,m} D\left(v^i \frac{\partial}{\partial x^i}, w^j \frac{\partial}{\partial x^j} \right) \\ &= \sum_{i,j}^{1,m} v^i D\left(\frac{\partial}{\partial x^i}, w^j \frac{\partial}{\partial x^j} \right) \\ &= \sum_{i,j}^{1,m} \left[v^i \frac{\partial w^j}{\partial x^i} \frac{\partial}{\partial x^j} + v^i w^j D\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \right] \\ &= \sum_{i,j}^{1,m} v^i \frac{\partial w^j}{\partial x^i} \frac{\partial}{\partial x^j} + \sum_{i,j,k}^{1,m} \Gamma^k_{ij} v^i w^j \frac{\partial}{\partial x^k}. \end{aligned}$$

Then, by exchanging $\boldsymbol{V}$ and $\boldsymbol{W}$ we also obtain

$$D(\boldsymbol{W}, \boldsymbol{V}) = \sum_{i,j}^{1,m} w^j \frac{\partial v^i}{\partial x^j} \frac{\partial}{\partial x^i} + \sum_{i,j,k}^{1,m} \Gamma^k_{ji} v^i w^j \frac{\partial}{\partial x^k},$$

so that

$$\begin{aligned} D(\boldsymbol{V}, \boldsymbol{W}) - D(\boldsymbol{W}, \boldsymbol{V}) &= \\ &= \sum_{i,j}^{1,m} \left[v^i \frac{\partial w^j}{\partial x^i} \frac{\partial}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \frac{\partial}{\partial x^i} \right] + \\ &\quad + \sum_{i,j,k}^{1,m} \left(\Gamma^k_{ji} - \Gamma^k_{ji} \right) v^i w^j \frac{\partial}{\partial x^k} \\ &= \sum_{i,j}^{1,m} \left[v^i \frac{\partial w^j}{\partial x^i} \frac{\partial}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \frac{\partial}{\partial x^i} \right] \end{aligned} \tag{10.2}$$

since by the assumptions, $\Gamma^k_{ij} = \Gamma^k_{ji}$. We now compute the commutator, remembering in the first step result *1.* of proposition 8.4:

$$\begin{aligned}
[\boldsymbol{V}, \boldsymbol{W}] &= \left[\sum_i^{1,m} v^i \frac{\partial}{\partial x^i}, \sum_j^{1,m} w^j \frac{\partial}{\partial x^j}\right] \\
&= \sum_{i,j}^{1,m} v^i w^j \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] + \\
&\qquad + \sum_{i,j}^{1,m} v^i \frac{\partial w^j}{\partial x^i}\frac{\partial}{\partial x^j} - \sum_{i,j}^{1,m} w^j \frac{\partial v^i}{\partial x^j}\frac{\partial}{\partial x^i} \\
&= \sum_{i,j}^{1,m} \left[v^i \frac{\partial w^j}{\partial x^i}\frac{\partial}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}\frac{\partial}{\partial x^i}\right] \qquad (10.3)
\end{aligned}$$

The first term in the equation before the last vanishes since we are in a coordinate basis and we thus see from (10.2) and (10.3) that

$$D(\boldsymbol{V}, \boldsymbol{W}) - D(\boldsymbol{W}, \boldsymbol{V}) = [\boldsymbol{V}, \boldsymbol{W}],$$

i.e. the connection is symmetric.

This completes the proof.

□

10.1.2 Smooth curves and covariant derivative along a curve

Definition 10.1 (Smooth curve on a manifold)
Let us consider a manifold $(\mathscr{M}, \mathcal{F})$. A smooth curve on $\mathscr{M}$ is a differentiable map

$$\sigma : [a, b] \longrightarrow \mathscr{M}$$

such that $\sigma(t) \in \mathscr{M}$. The tangent vector to the curve is denoted by $\dot{\boldsymbol{\sigma}}(t)$, which is defined as

$$\dot{\boldsymbol{\sigma}}(t) = d\sigma\rceil_t \left(\left.\frac{d}{dr}\right\rceil_t\right).$$

Remember that the differential of $\sigma(t)$ is a map

$$d\sigma\rceil_t : \mathbb{R}_t \cong \mathbb{R} \longrightarrow \mathscr{M}_{\sigma(t)},$$

which maps tangent vectors in $\mathbb{R}_t$ into tangent vectors of $\mathscr{M}_{\sigma(t)}$. This helps us in giving a precise characterization of the tangent vector $\dot{\boldsymbol{\sigma}}(t)$. Indeed let us consider a coordinate neighborhood (U, ϕ) on $\mathscr{M}$, where ϕ is associated to the coordinates $(x^1, \dots, x^m)$. We fix as usual the coordinate basis on the tangent spaces of points in U. The components of the vector $\dot{\boldsymbol{\sigma}}(t)$ (which is a map from $C^\infty(\mathscr{M})$ into $\mathbb{R}$) are

$$(\dot{\boldsymbol{\sigma}}(t))(x^i) = \left(d\sigma\rceil_t \left(\left.\frac{d}{dr}\right\rceil_t\right)\right)(x^i) = \left.\frac{d}{dr}\right\rceil_t (x^i \circ \sigma) = \left.\frac{d\sigma^i}{dr}\right\rceil_t \mathrel{\hat{=}} \frac{d\sigma^i(t)}{dt} \quad ,$$

where $\sigma^i = x^i \circ \sigma$ is the i-th coordinate component of the map σ defining the curve. We can thus write

$$\dot{\boldsymbol{\sigma}}(t) = \sum_i^{1,m} \frac{d\sigma^i}{dr}\bigg]_t \frac{\partial}{\partial x^i}\bigg]_{\sigma(t)} \mathrel{\mathring{=}} \sum_i^{1,m} \frac{d\sigma^i(t)}{dt}\frac{\partial}{\partial x^i}\bigg]_{\sigma(t)}.$$

In what follows we are also going to use the notation $x^i(t)$ in place of $\sigma^i(t)$ for the components of the curve.

Proposition 10.2 (Covariant derivative along a curve)
Let $\sigma(t) : [a, b] \longrightarrow \mathscr{M}$ be a differentiable curve on a manifold $(\mathscr{M}, \mathcal{F})$ with connection $D(-,-)$. Let $\boldsymbol{V}(t)$ be a differentiable vector field along σ. There exists one and only one map which associates to a vector field $\boldsymbol{V}$ along σ another vector field $D\boldsymbol{V}/dt$ along σ, the covariant derivative of $\boldsymbol{V}$ along σ*, such that:*

1. $\dfrac{D(\boldsymbol{V}+\boldsymbol{W})}{dt} = \dfrac{D\boldsymbol{V}}{dt} + \dfrac{D\boldsymbol{W}}{dt}$;
2. $\forall f : [a, b] \longrightarrow \mathbb{R}$ *we have* $\dfrac{D(f\boldsymbol{V})}{dt} = \dfrac{df}{dt}\boldsymbol{V} + f\dfrac{D\boldsymbol{V}}{dt}$;
3. *if* $\boldsymbol{Y} \in \mathcal{V}(\mathscr{M})$ *is a vector field on* $\mathscr{M}$ *such that* $\boldsymbol{V}(t) = Y(\sigma(t))$ *then*
$$\frac{D\boldsymbol{V}}{dt} = D\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{Y}\right)_{\sigma(t)}. \qquad (10.4)$$

Proof:

Let us choose a chart $(U, \phi) \in \mathcal{F}$ on the manifold $(\mathscr{M}, \mathcal{F})$ with coordinate functions $(x^1, \ldots, x^m)$. and let us consider a curve $\sigma(t) = (x^1(t), \ldots, x^m(t))$. We then have

$$\dot{\boldsymbol{\sigma}}(t) = \sum_i^{1,m} \frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i}.$$

To prove the existence we use property *3.* as an ansatz, i.e. we define

$$\frac{D\boldsymbol{V}}{dt} \overset{\text{def.}}{=} D\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{Y}\right)_{\sigma(t)};$$

this is a good definition since the operation defined by the connection is local, i.e. it depends only on the values of the vector fields at a given point and thus it makes sense for each vector field which is defined at that point. Moreover we have:

$$\begin{aligned}
\frac{D(\boldsymbol{V}+\boldsymbol{W})}{dt} &= D\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{V}+\boldsymbol{W}\right)_{\sigma(t)} \\
&= D\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{V}\right)_{\sigma(t)} + D\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{W}\right)_{\sigma(t)} \\
&= \frac{D\boldsymbol{V}}{dt} + \frac{D\boldsymbol{W}}{dt},
\end{aligned}$$

so that *1.* is satisfied. Then we have

$$\begin{aligned}
\frac{D(f\boldsymbol{V})}{dt} &= D\left(\dot{\boldsymbol{\sigma}}(t), f\boldsymbol{V}\right)_{\sigma(t)} \\
&= (\dot{\boldsymbol{\sigma}}(t))(f)\boldsymbol{V} + fD\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{V}\right)_{\sigma(t)} \\
&= \sum_i^{1,m} \frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i}(f) + fD\left(\dot{\boldsymbol{\sigma}}(t), \boldsymbol{V}\right)_{\sigma(t)} \\
&= \frac{df}{dt}\boldsymbol{V} + f\frac{D\boldsymbol{V}}{dt}
\end{aligned}$$

and *2.* is also satisfied. *3.*, of course, holds by definition, so the only property we still have to prove is uniqueness. To establish it we rewrite $D\boldsymbol{V}/dt$ using the local expression above for $\dot{\boldsymbol{\sigma}}(t)$ and also writing locally the vector field along $\sigma(t)$ as

$$\boldsymbol{V}(t) = \sum_{j}^{1,m} v^j(t) \frac{\partial}{\partial x^j}.$$

Then from equation (10.4) we can obtain the following chain of equalities:

$$\begin{aligned}
\frac{D\boldsymbol{V}}{dt} &= \frac{D\left(\sum_j^{1,m} v^j(t)(\partial/\partial x^j)\right)}{dt} \\
&= \sum_j^{1,m} \left(\frac{dv^j(t)}{dt}\frac{\partial}{\partial x^j} + v^j(t)\frac{D(\partial/\partial x^j)}{dt} \right) \\
&= \sum_j^{1,m} \left[\frac{dv^j(t)}{dt}\frac{\partial}{\partial x^j} + v^j(t) D\left(\dot{\boldsymbol{\sigma}}(t), \frac{\partial}{\partial x^j} \right) \right] \\
&= \sum_j^{1,m} \left[\frac{dv^j(t)}{dt}\frac{\partial}{\partial x^j} + v^j(t) D\left(\sum_i^{1,m} \frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \right] \\
&= \sum_k^{1,m} \frac{dv^k(t)}{dt}\frac{\partial}{\partial x^k} + \sum_{i,j}^{1,m} v^j(t)\frac{dx^i(t)}{dt} D\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \\
&= \sum_k^{1,m} \frac{dv^k(t)}{dt}\frac{\partial}{\partial x^k} + \sum_{i,j}^{1,m} v^j(t)\frac{dx^i(t)}{dt}\Gamma^k_{ij}\frac{\partial}{\partial x^k} \\
&= \sum_k^{1,m} \left(\frac{dv^k(t)}{dt} + \sum_{i,j}^{1,m} \Gamma^k_{ij}\frac{dx^i(t)}{dt} v^j(t) \right) \frac{\partial}{\partial x^k}. \qquad (10.5)
\end{aligned}$$

We thus see that the covariant derivative along $\sigma(t)$ is completely determined by the connection coefficients in a unique way, i.e., given the connection, it is unique. This completes the proof.

□