Chapter 8

Lecture 8

8.1 Some reminders of topology

We are going to recall here some basic definitions in topology that are well known from other courses. This will give us the opportunity to set up the notation. We start with the definition of

Definition 8.1 (Topology and open sets)
Let $\mathscr{S}$ be a set and $\mathcal{T}$ a collection of subsets of $\mathscr{S}$ such that:

1. $\mathscr{S} \in \mathcal{T}$ *and* $\emptyset \in \mathcal{T}$;
2. *given* $n \in \mathbb{N}$, $A_i \in \mathcal{T}$, $i = 1, \ldots, n \Rightarrow \bigcap_i^{1,n} A_i \in \mathcal{T}$;
3. *given a collection* $\{A_n\}_{n\in\mathbb{N}}$, $A_n \in \mathcal{T}$ $\forall n \in \mathbb{N} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{T}$.

$\mathcal{T}$ *is called a* topology *on* $\mathscr{S}$; *its elements are called* open sets.

A topological space is then a space that has a topology:

Definition 8.2 (Topological space)
Let $\mathscr{S}$ be a set and $\mathcal{T}$ a topology on $\mathscr{S}$. The couple $(\mathscr{S}, \mathcal{T})$ is a topological space.

In a topological space we can intuitively define the concept of *beeing close* to a given point of the topological space. This motivates the following

Definition 8.3 (Neighborhood)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space and $\mathtt{p} \in \mathscr{S}$. A neighborhood *of $\mathtt{p}$ is an open set $P \in \mathcal{T}$ such that $\mathtt{p} \in P$.*

To characterize topological spaces and their property an important concept is that of a collection of sets called cover:

Definition 8.4 (Cover)
Let $\mathscr{S}$ be a set and $\mathscr{U} = \{S_\alpha\}_{\alpha\in\mathcal{A}}$ a collection of subsets of $\mathscr{S}$ indexed by a set $\mathcal{A}$. $\mathscr{U}$ is called a cover *of $\mathscr{S}$ if $\bigcup_{\alpha\in\mathcal{A}} S_\alpha = \mathscr{S}$.*

A subcover will be defined according to an intuitive idea as follows:

Definition 8.5 (Subcover)
Let $\mathscr{S}$ be a set and $\mathscr{U} = \{S_\alpha\}_{\alpha\in\mathcal{A}}$ a cover of $\mathscr{S}$. Let $\mathcal{A}' \subseteq \mathcal{A}$. Then $\mathscr{U}' = \{S_{\alpha'}\}_{\alpha'\in\mathcal{A}'}$ such that $\bigcup_{\alpha'\in\mathcal{A}'} S_{\alpha'} = \mathscr{S}$ is a subcover *of the cover $\mathscr{U}$ of $\mathscr{S}$.*

Of course, a subcover is itself a cover. A distinction between covers can be made according to the following definitions.

Definition 8.6 (Refinement)
Let $\mathscr{S}$ be a set and $\mathscr{U} = \{S_\alpha\}_{\alpha\in\mathcal{A}}$ a cover of $\mathscr{S}$. Another cover $\mathscr{V} = \{S'_\beta\}_{\beta\in\mathcal{B}}$ of $\mathscr{S}$ is called a refinement *of $\mathscr{U}$ if $\forall\beta\in\mathcal{B}$, $\exists\alpha\in\mathcal{A}$ such that $S'_\beta \subset S_\alpha$.*

Definition 8.7 (Open cover)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space and let $\mathcal{O} = \{O_\alpha\}_{\alpha\in\mathcal{A}}$ be a cover of $\mathscr{S}$. $\mathcal{O}$ is open cover *of $\mathscr{S}$ if $S_\alpha \in \mathcal{T}$ $\forall\alpha\in\mathcal{A}$.*

Definition 8.8 (Locally finite open cover)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space and $\mathcal{O} = \{O_\alpha\}_{\alpha\in\mathcal{A}}$ an open cover of $\mathscr{S}$. $\mathcal{O}$ is a locally finite open cover *of $\mathscr{S}$ if $\forall \mathfrak{s} \in \mathscr{S}$ there exists W, open neighborhood of $\mathfrak{s}$, such that $\{O_i | O_i \cap W \neq \emptyset\}$ is a finite set.*

In terms of the above definitions help us in defining two special kinds of topological spaces.

Definition 8.9 (Compact topological space)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space. $\mathscr{S}$ is compact *if every open cover of $\mathscr{S}$ admits a finite subcover.*

Definition 8.10 (Paracompact topological space)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space. $\mathscr{S}$ is paracompact *if every open cover of $\mathscr{S}$ admits a locally finite open refinement.*

A third kind, that is at the heart of the concept of a differentiable manifold is a space of the Hausdorff kind.

Definition 8.11 (Hausdorff topological space)
Let $(\mathscr{S}, \mathcal{T})$ be a topological space. $\mathscr{S}$ is a Hausdorff space *if $\forall \mathfrak{p}, \mathfrak{q} \in \mathscr{S}$ there exist P and Q, open neighborhoods of $\mathfrak{p}$ and $\mathfrak{q}$ respectively, such that $P \cap Q = \emptyset$.*

We are going to use some of the concepts in the last definitions shortly, when dealing with partitions of unity. First we collect dome further results in differential geometry.

8.2 Some reminders of differential geometry

We are now going to apply the concept of tensor in the context of differentiable manifolds, to define tensor fields. In mathematical physics tensor fields are our variables, for the theory we develop, and play the same role of the *scalar* field that we have *heuristically* introduced in the second lecture. By the way a scalar field will be nothing but an $(0, 0)$ tensor. In this lecture we will also set up the notation for some already known concepts in differential geometry and topology.

Figure 8.1: Typical example of a non-Hausdorff topological space.

8.2.1 Vector bundles and sections

We will recall in what follows the concept of vector bundle, before applying it to the definition of tensor fields on manifolds.

Definition 8.12 (Vector bundle)
Let $\mathscr{M}$ and $\mathscr{B}$ be two manifolds, V a vector space and $\pi : \mathscr{B} \longrightarrow \mathscr{M}$ a differential map such that:

1. *π is surjective;*
2. *$\forall \mathsf{m} \in \mathscr{M}$ there exists $U \subset \mathscr{M}$ neighborhood of m such that $\pi^{-1}(U)$ is isomorphic with $U \times V$.*

Then $\mathscr{B}$ is called a vector bundle *over $\mathscr{M}$.*

$\mathscr{M}$ is called the *base space*, V is the *fiber* (see figure 8.2 for a schematic representation).

Definition 8.13 (Section of a vector bundle)
Let $\mathscr{B}$ be a vector bundle over $\mathscr{M}$. A map $\Sigma : \mathscr{M} \longrightarrow \mathscr{B}$ such that

$$\pi \circ \Sigma = \mathbb{I}_{\mathscr{M}}$$

is called a section *of $\mathscr{B}$.*

In figure 8.3 there is a graphical representation of a section.

8.2.2 Partition of unity

Definition 8.14 (Differentiable partition of unity)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold. A differentiable partition of unity *is a couple $(\mathscr{R}, \mathcal{P})$ where:*

1. *$\mathscr{R}$ is a locally finite open cover of $\mathscr{M}$;*

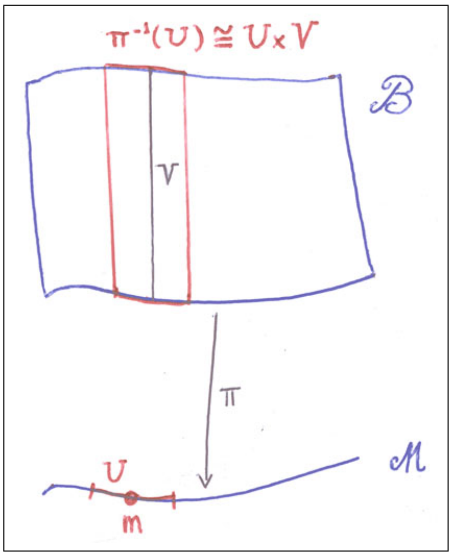

Figure 8.2: Vector bundle.

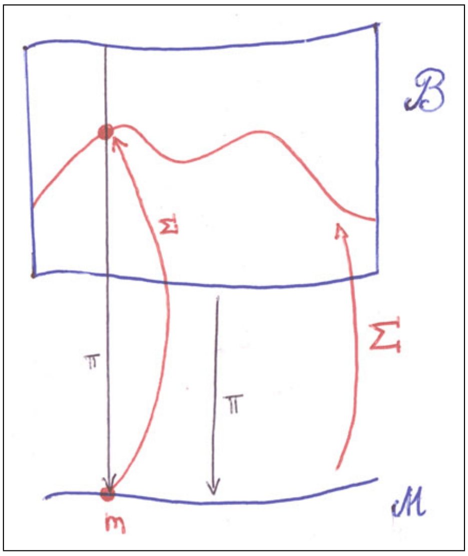

Figure 8.3: Section of a vector bundle.

2. $\mathcal{P}$ is a collection of functions

$$\mathcal{P} = \{f_V : \mathscr{M} \longrightarrow \mathbb{R} | V \in \mathscr{R},\ f \text{ differentiable}\} \quad \text{such that}$$

(a) $f_V \geq 0,\ \forall V \in \mathscr{R}$;

(b) $\operatorname{supp}(f_V) \subset V$;

(c) $\sum_{V \in \mathscr{R}} f_V = 1$.

We see that the sum is finite because $\mathscr{R}$ is a locally finite open cover of $\mathscr{M}$. Thus $\forall \mathtt{m} \in \mathscr{M}$ it is possible to find a neighborhood P which intersects only a finite number of $V \in \mathscr{R}$. In that neighborhood the sum is thus restricted only to this finite number V's.

Proposition 8.1 (Existence of partition of unity)
Let $(\mathscr{M}, \mathcal{F})$ be a paracompact differentiable manifold and let $\mathscr{U}$ be an open cover of $\mathscr{M}$. There exists a partition of unity $(\mathscr{R}, \mathcal{P})$ where $\mathscr{R}$ is a locally finite open refinement of $\mathscr{U}$.

We will say that the partition of unity $(\mathscr{R}, \mathcal{P})$ is subordinated to the cover $\mathscr{U}$. The paracompactness is required to obtain the open locally finite refinement $\mathscr{R}$ starting from $\mathscr{U}$.

8.3 Tensors - 3 -

8.3.1 Tensors (Tensor Fields) on Manifolds

As it has been proved in the course of differential geometry if we consider the set $T(\mathscr{M}) = \bigcup_{\mathtt{m} \in \mathscr{M}} \mathscr{M}_{\mathtt{m}}$, it has a natural differentiable manifold structure. The same is true for $T^r_s(\mathscr{M}) \stackrel{\text{def.}}{=} \bigcup_{\mathtt{m} \in \mathscr{M}} T^r_s(\mathscr{M}_{\mathtt{m}})$ where $T^r_s(\mathscr{M}_{\mathtt{m}})$ is the vector space of tensors of the type (r, s) on the vector space $\mathscr{M}_{\mathtt{m}}$, i.e. on the tangent space of $\mathscr{M}$ in $\mathtt{m}$. We give then the following definition.

Definition 8.15 (Tensor bundle of the (r, s) type)
Let $(\mathscr{M}, \mathcal{F})$ be a manifold and

$$T^r_s(\mathscr{M}) \stackrel{\text{def.}}{=} \bigcup_{m \in \mathscr{M}} T^r_s(\mathscr{M}_m).$$

$T^r_s(\mathscr{M})$ together with the canonical projection

$$\pi^r_s : T^r_s(\mathscr{M}) \longrightarrow \mathscr{M}$$

*is the (r, s)-*tensor bundle *over $\mathscr{M}$.*

The tensor bundle is a vector bundle: its base space is the manifold $\mathscr{M}$, its fiber is the tensor product $T^r_s(\mathscr{M}_{\mathtt{m}})$. The canonical projection in a given chart (U, ϕ) is the map that associates to a point $\mathtt{p}_T$ in the tensor bundle the point $\mathtt{m}$ associated to the fiber to which the point $\mathtt{p}_T$ belongs. In symbols, $\forall \mathtt{p}_T \in T^r_s(\mathscr{M})$ then $\pi(\mathtt{p}_T) = \mathtt{m}$ if $\mathtt{p}_T \in T^r_s(\mathscr{M}_{\mathtt{m}})$. A natural way to define coordinates on the tensor bundle is to associate to each point $\mathtt{p}_T$ in the tensor bundle:

1. the coordinates of the point $\mathtt{m} \in \mathscr{M}$ associated to the fiber to which the point $\mathtt{p}_T$ belongs ...
2. ... **and** the components of the tensor associated to $\mathtt{p}_T$ in $T^r_s(\mathscr{M}_\mathtt{m})$.

This gives in a straightforward way an atlas of $T^r_s(\mathscr{M})$. The concept of a tensor at a point can be extended to that of a (smooth) tensor field over $\mathscr{M}$.

Definition 8.16 (Smooth tensor field)
A smooth tensor field *over $\mathscr{M}$ is a section of $T^r_s(\mathscr{M})$. We will denote the space of all tensor fields of the (r,s) type on $\mathscr{M}$ with $\mathcal{T}^r_s(\mathscr{M})$.*

Some concepts already developed in the course of differential geometry are special cases of the definitions above. In particular

Definition 8.17 (Tangent and Cotangent bundle)
The tangent and cotangent bundles over a manifold $\mathscr{M}$ are the tensor bundles $T^1_0(\mathscr{M})$ and $T^0_1(\mathscr{M})$ respectively.

Definition 8.18 (Vector fields and 1-form fields)
A smooth vector field *is a tensor field of the type $(1,0)$ and a* smooth 1-form field *is a tensor field of the type $(0,1)$.*

Moreover we set up the following additional notations:

Notation 8.1 (Particular cases of bundles and spaces of fields)
We will use the following notations:

$T(\mathscr{M})$	for	the tangent bundle over $\mathscr{M}$
$T^*(\mathscr{M})$	for	the cotangent bundle over $\mathscr{M}$
$\mathcal{V}(\mathscr{M})$	for	the space of all vector fields over $\mathscr{M}$
$\mathcal{E}(\mathscr{M})$	for	the space of all $1-$form fields over $\mathscr{M}$

For later convenience we also add at this point a definition outside our main discussion:

Definition 8.19 (Line element field)
A line element field *over $\mathscr{M}$ is is a section of the line bundle over $\mathscr{M}$, i.e. it is a smooth assignment of a couple $(\boldsymbol{v}, -\boldsymbol{v})$ with $\boldsymbol{v} \in \mathscr{M}_\mathtt{m}$ at all $\mathtt{m} \in \mathscr{M}$.*

Differentiable tensor fields can be characterized in terms of some equivalent properties according to the following proposition.

Proposition 8.2 (Characterization of smooth tensor fields)
Let $\boldsymbol{T}$ be a tensor field of the (r,s) type on an open subset $W \subset \mathscr{M}$. The following conditions are equivalent:

1. *$\boldsymbol{T}$ is differentiable;*
2. *given a chart $(U,\phi) \in \mathcal{F}$ with coordinate functions $(x^1, \ldots, x^m)$ if we consider*

$$\boldsymbol{T}\rceil_U = \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots,j_s}}^{1,m} T^{i_1\ldots i_r}_{j_1\ldots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s},$$

then $T^{i_1\ldots i_r}_{j_1\ldots j_s} : U \subset \mathscr{M} \longrightarrow \mathbb{R}$ are differentiable functions on U.

To understand the notation in point *2.* above, remember that at each point $\mathtt{m}$ a coordinate basis of $\mathscr{M}_{\mathtt{m}}$ is $\{\partial/(\partial x^i)\}_{i=1,\ldots,m}$ and the corresponding dual basis in $\mathscr{M}^*_{\mathtt{m}}$ is $\{dx^i\}_{i=1,\ldots,m}$. Then by generalizing proposition 7.2 we have that

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}, \quad \forall (i_1, \ldots, i_r) \text{ extracted from } \{1, \ldots, n\} \text{ and } \forall (j_1, \ldots, j_s) \text{ extracted from } \{1, \ldots, n\} \right\}$$

is a basis of $T^r_s(\mathscr{M})$. The *functions* $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ are the components of $\boldsymbol{T}$ in the given basis[1]. We are going to prove a particular version of the above proposition (with an additional result) concerning vector fields. This proposition can be generalized to tensor fields, where only the notation is slightly more cumbersome.

Proposition 8.3 (Characterization of smooth vector fields)
Let $\boldsymbol{X}$ be a vector field on an open subset $W \subset \mathscr{M}$. The following properties are equivalent:

1. *$\boldsymbol{X}$ is differentiable;*
2. *given a chart $(U, \phi) \in \mathcal{F}$ with coordinate functions $(x^1, \ldots, x^m)$ if we consider*
$$\boldsymbol{X}\rceil_U = \sum_i^{1,m} X^i \frac{\partial}{\partial x^i},$$
then $X^i : U \subset \mathscr{M} \longrightarrow \mathbb{R}$ are differentiable functions on U;
3. *if $V \subset \mathscr{M}$ is open and $f \in C^\infty(V)$, then $\boldsymbol{X}(f) \in C^\infty(V)$, where we define*
$$\boldsymbol{X}(f)(\mathtt{m}) \stackrel{\text{def.}}{=} \boldsymbol{X}_{\mathtt{m}}(f),$$
i.e. $\boldsymbol{X}_{\mathtt{m}}$ is the vector $\boldsymbol{X}(\mathtt{m})$.

Proof:

$1 \Rightarrow 2$ If $\boldsymbol{X}$ is differentiable then given a coordinate system (U, ϕ) then $\boldsymbol{X}\rceil_U$,
$$\boldsymbol{X}\rceil_U : U \longrightarrow T(\mathscr{M})$$
is differentiable. Moreover, since x^i is a coordinate function, $dx^i \circ \boldsymbol{X}\rceil_U$ is differentiable. But $dx^i \circ \boldsymbol{X}\rceil_U = X^i$ on U and the proof is complete.

[1] Often we are going to use the terminology *in the given coordinate system* or *in the given reference frame* when we consider the coordinate basis for tangent and cotangent space associated to a given chart on the manifold. **It is important to stress that** *non-coordinate basis* **can be chosen as well!!! Not** all results valid in coordinate basis for the components of tensor are valid in non-coordinate basis**!!!**

2 ⇒ 3 On an open set V let us consider $f \in C^\infty(V)$. Let (U, ϕ) be a coordinate system on $\mathscr{M}$. Then, denoting by π the canonical projection of the tangent bundle,

$$\boldsymbol{X}(f) = \sum_i^{1,m} X^i \frac{\partial f}{\partial x^i}$$

is such that the X^i are differentiable functions by hypothesis and $\partial f / \partial x^i$ is differentiable since f is C^∞; thus $\boldsymbol{X}(f)$ is also differentiable, as stated.

3 ⇒ 1 Let (U, ϕ) be a coordinate system on $\mathscr{M}$ chosen arbitrarily and let us call $(x^1, \ldots, x^m)$ the coordinate functions on U. Then

$$(x^1(\pi(\boldsymbol{v})), \ldots, x^m(\pi(\boldsymbol{v})), dx^1(\boldsymbol{v}), \ldots, dx^m(\boldsymbol{v}))$$

is a coordinate system on $T(\mathscr{M})$, i.e. it gives coordinates for each $\boldsymbol{v} \in \mathscr{M}_{\mathtt{m}}$ with $\mathtt{m} \in U$. Thus the differentiability of $x^i \circ \pi \circ \boldsymbol{X}\rceil_U = x^i$ and of $dx^i \circ \boldsymbol{X}\rceil_U = \boldsymbol{X}(x^i)$ (which is implied by 3. with $f = x^i$) yields the differentiability of $\boldsymbol{X}\rceil_U$.

The proof is thus complete.

□

On couples of vector fields there is an important application defined, the *Lie Brackets*. In the following we give its definition as well as its properties.

Definition 8.20 (Lie Brackets)
Let us consider two vector fields $\boldsymbol{X}$, $\boldsymbol{Y} \in \mathcal{V}(\mathscr{M})$. *The map*

$$[-,-] : \mathcal{V}(\mathscr{M}) \times \mathcal{V}(\mathscr{M}) \longrightarrow \mathcal{V}(\mathscr{M})$$

which associates to $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *the vector field* $[\boldsymbol{X}, \boldsymbol{Y}]$ *defined as*

$$[\boldsymbol{X}, \boldsymbol{Y}]_{\mathtt{m}}(f) \stackrel{\text{def.}}{=} \boldsymbol{X}_{\mathtt{m}}(\boldsymbol{Y}(f)) - \boldsymbol{Y}_{\mathtt{m}}(\boldsymbol{X}(f))$$

where

$$[\boldsymbol{X}, \boldsymbol{Y}]_{\mathtt{m}} = [\boldsymbol{X}, \boldsymbol{Y}](\mathtt{m}) \in \mathscr{M}_{\mathtt{m}}.$$

We quickly comment about the well-definiteness of the above definition. In particular we observe that $[\boldsymbol{X}, \boldsymbol{Y}]$ is a vector field, let us say $\boldsymbol{Z}$. Thus it is a section of the tangent bundle $T(\mathscr{M})$ and it associates to each $\mathtt{m} \in \mathscr{M}$ a vector in the tangent space $\mathscr{M}_{\mathtt{m}}$; we could denote this vector as $\boldsymbol{Z}(\mathtt{m})$, but we are going to use the notation $\boldsymbol{Z}_{\mathtt{m}}$, i.e. the notation $[\boldsymbol{X}, \boldsymbol{Y}]_{\mathtt{m}}$, as explicitly said in the last line of the above definition. With this notation we make explicit that $[\boldsymbol{X}, \boldsymbol{Y}]_{\mathtt{m}} \in \mathscr{M}_{\mathtt{m}}$. Thus $[\boldsymbol{X}, \boldsymbol{Y}]_{\mathtt{m}}$, being a tangent vector, maps a (germ of) function(s) at $\mathtt{m} \in \mathscr{M}$ into $\mathbb{R}$. To properly define its action we have to define the result of the operation $[\boldsymbol{X}, \boldsymbol{Y}]_{\mathtt{m}}(f)$ in terms of the vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$. To this end let us observe again that if $\boldsymbol{Y} \in \mathcal{V}(\mathscr{M})$ then given $f \in C^\infty(\mathscr{M})$, then $\boldsymbol{Y}(f) \in C^\infty(\mathscr{M})$. Thus the tangent vector $\boldsymbol{X}_{\mathtt{m}}$ transforms the (germ of) function(s) $\boldsymbol{Y}(f)$ into a real number, $\boldsymbol{X}_{\mathtt{m}}(\boldsymbol{Y}(f)) \in \mathbb{R}$. The same is true if we exchange the roles of $\boldsymbol{X}$ and $\boldsymbol{Y}$, which shows that the definition given above is consistent. With the following proposition we are going to add some details to the definition.

Proposition 8.4 (Properties of the Lie Brackets)
The Lie brackets have the following properties:

1. $\forall f, g \in C^\infty(\mathscr{M})$ *we have*

$$[f\boldsymbol{X}, g\boldsymbol{Y}] = fg\,[\boldsymbol{X}, \boldsymbol{Y}] + f\boldsymbol{X}(g)\boldsymbol{Y} - g\boldsymbol{Y}(f)\boldsymbol{X};$$

2. *it is antisymmetric, i.e.* $[\boldsymbol{X}, \boldsymbol{Y}] = -[\boldsymbol{Y}, \boldsymbol{X}]$*;*

3. *it satisfies the Jacobi identity, i.e.*

$$[[\boldsymbol{X}, \boldsymbol{Y}], \boldsymbol{Z}] + [[\boldsymbol{Y}, \boldsymbol{Z}], \boldsymbol{X}] + [[\boldsymbol{Z}, \boldsymbol{X}], \boldsymbol{Y}] = 0.$$

We will comment property *1.* just to make explicit that it is a well defined expression. In particular we observe that a function $f \in C^\infty(\mathscr{M})$ times a vector field $\boldsymbol{X}$ is again a vector field. Thus the left-hand side in *1.* is properly defined. On the right-hand side we quickly discuss the second term. With $f, g \in C^\infty(\mathscr{M})$, since $\boldsymbol{X}(g) \in C^\infty(\mathscr{M})$, then the product $f\boldsymbol{X}(g) \in C^\infty(\mathscr{M})$. Thus $f\boldsymbol{X}(g)\boldsymbol{Y} \in \mathcal{V}(\mathscr{M})$; the same is then true for the third term and *a fortiori* for the first one, so the expression in *1.* is meaningful.