Chapter 6

Lecture 6

6.1 Tensors - 1 -

In this and the next lecture let V, W, U be finite dimensional vector spaces over a field $\mathbb{F}$ (for definiteness $\mathbb{F}$ can be taught as $\mathbb{R}$ or $\mathbb{C}$). We are going to define what is the tensor product of vector space, to define what are tensors and to prove (or state) some of their property. Our final goal will be to show how, in the case of the multiple tensor product of a space and its dual, tensors can be identified with multilinear applications. Almost all the proofs will rely upon a crucial property of the tensor product, the universal factorization property: it gives us a mean to transfer properties of (multi)linear maps (from the cartesian product of vector spaces in the field $\mathbb{F}$) into properties of maps from the tensor product of the same vector spaces (again into the field $\mathbb{F}$).

6.1.1 Tensor product

Let $F(V,W)$ be the *free vector space* generated by all couples (v,w) with $v \in V$ and $w \in W$: thus $F(V,W)$ is the set of all finite linear combinations of couples (v,w). $R(V,W)$ will be the subspace of $F(V,W)$ spanned by the following elements:

$$\begin{aligned}
(v_1+v_2,w)-(v_1,w)-(v_2,w) &\qquad v_1,v_2\in V,\quad w\in W\\
(v,w_1+w_2)-(v,w_1)-(v,w_2) &\qquad v\in V,\quad w_1,w_2\in W\\
(\alpha v,w)-\alpha(v,w) &\qquad v\in V,\quad w\in W,\quad \alpha\in\mathbb{F}\\
(v,\alpha w)-\alpha(v,w) &\qquad v\in V,\quad w\in W,\quad \alpha\in\mathbb{F}
\end{aligned}$$

Definition 6.1 (Tensor product)
The tensor product *of two vector spaces V and W is the vector space $V\otimes W$ defined as*

$$V\otimes W \stackrel{\text{def.}}{=} F(V,W)\setminus R(V,W) \quad .$$

The equivalence class in $V\otimes W$ containing the element (v,w) is denoted as $v\otimes w$. We will call ϕ the canonical bilinear map

$$\phi : V\times W \longrightarrow V\otimes W$$

such that $\phi(v, w) = v \otimes w$.

Definition 6.2 (Universal factorization property)
Let ψ *be a bilinear map*

$$\psi : V \times W \longrightarrow U$$

We will say that the couple (U, ψ) *has the* universal factorization property *for* $V \times W$*, if* $\forall S$*,* S *vector space, and*

$$\forall f, f : V \times W \longrightarrow S$$

f *bilinear, there exists a unique* $\tilde{f}$

$$\tilde{f} : U \longrightarrow S$$

such that $f = \tilde{f} \circ \psi$.

Proposition 6.1 (Tensor product: universal factorization)
The couple $(V \otimes W, \phi)$ *has the universal factorization property for* $V \times W$*. Moreover the couple* $(V \otimes W, \phi)$ *is unique in the sense that if another couple* (Z, ζ) *has the universal factorization property for* $V \times W$*, then there exists an isomorphism* α

$$\alpha : V \otimes W \longrightarrow Z$$

such that $\zeta = \alpha \circ \phi$.

Proof:

Let S be any vector space and f a bilinear map

$$f : V \times W \longrightarrow S$$

Since $V \times W$ is a basis for $F(V, W)$, f can be extended by linearity to a unique map

$$f' : F(V, W) \longrightarrow S$$

by the rule

$$f'\Big(\sum_i^{1,N} \lambda_i (v_i, w_i)\Big) = \sum_i^{1,N} \lambda_i f(v_i, w_i).$$

Since f is bilinear $ker(f') \supset R(V, W)$[1]. This means that f' induces a well defined map f''

$$f'' : V \otimes W \longrightarrow S$$

[1] To understand this fact consider for example the action of f' on an element of the form $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$. We have

$$\begin{aligned} f'((v_1 + v_2, w) - (v_1, w) - (v_2, w)) &= f'((v_1 + v_2, w)) - f'((v_1, w)) - f'((v_2, w)) \\ &= f(v_1 + v_2, w) - f(v_1, w) - f(v_2, w) \\ &= f(v_1, w) + f(v_2, w) - f(v_1, w) - f(v_2, w) \\ &= 0 \quad , \quad \forall v_1, v_2 \in V, \quad \forall w \in W \quad , \end{aligned} \tag{6.1}$$

where we used the bilinearity of f. With analogous calculations we see that f' vanishes on the other combinations that are used to span $R(V, W)$ so by linearity it vanishes on all $R(V, W)$.

such that[2] $f''(v \otimes w) = f'((v, w))$. By construction $f'' \circ \phi = f$ and f'' is unique since $\phi(V \times W)$ spans $V \otimes W$. This shows that the couple $(V \otimes W, \phi)$ has the universal factorization property for $V \times W$.
Let us consider another couple (Z, ζ) having the universal factorization property for $V \times W$. When in the definition of the universal factorization property we use the following identifications

$$\begin{array}{ll} \psi \longleftrightarrow \phi & U \longleftrightarrow V \otimes W \\ f \longleftrightarrow \zeta & S \longleftrightarrow Z \end{array}$$

we obtain the existence of a unique map σ_1,

$$\sigma_1 : V \otimes W \longrightarrow Z$$

such that $\zeta = \sigma_1 \circ \phi$.
At the same time we can exchange the roles of $(U \otimes V, \phi)$ and (Z, ζ). This means that in the definition of the universal factorization property we can also use the following identifications

$$\begin{array}{ll} \psi \longleftrightarrow \zeta & U \longleftrightarrow Z \\ f \longleftrightarrow \phi & S \longleftrightarrow V \otimes W \end{array}$$

so that it exists a unique map σ_2,

$$\sigma_2 : Z \longrightarrow V \otimes W$$

such that $\phi = \sigma_2 \circ \zeta$.
We thus have

$$\begin{array}{rcl} \zeta &=& \sigma_1 \circ \sigma_2 \circ \zeta \\ \phi &=& \sigma_2 \circ \sigma_1 \circ \phi \end{array}$$

and by the uniqueness of the map in the definition of the universal factorization property we obtain

$$\begin{array}{rcl} \sigma_1 \circ \sigma_2 &=& \mathbb{I}_Z \\ \sigma_2 \circ \sigma_1 &=& \mathbb{I}_{V \otimes W} \end{array}$$

so that Z and $V \otimes W$ are isomorphic.

□

6.1.2 Properties of tensor product

Proposition 6.2 (Isomorphism of $V \otimes W$ into $W \otimes V$)
There exists only one isomorphism of $V \otimes W$ onto $W \otimes V$ which $\forall v, w$ sends $v \otimes w$ into $w \otimes v$.

[2] This can be seen by writing the class $v \otimes w$ as $(v, w) + R(V, W)$. But then

$$f'((v, w) + R(V, W)) = f'((v, w)) + f'(R(V, W)) = f'((v, w)) + 0 = f'((v, w))$$

because we remember that $ker(f') \supset R(V, W)$.

Proof:

Let us consider the universal factorization property of $(V \otimes W, \phi_{VW})$ for $V \times W$ with respect to the map f

$$f : V \times W \longrightarrow W \otimes V$$

defined as $f(v, w) \stackrel{\text{def.}}{=} w \otimes v$. Then we know that there exists only one map f'' such that

$$f'' : V \otimes W \longrightarrow W \otimes V$$

and $f''(v \otimes w) = w \otimes v$.
At the same time we can consider the universal factorization property of $(W \otimes V, \phi_{WV})$ for $W \times V$ with respect to the map g

$$g : W \times V \longrightarrow V \otimes W$$

defined as $g(w, v) \stackrel{\text{def.}}{=} v \otimes w$. Then we know that there exists only one map g'' such that

$$g'' : W \otimes V \longrightarrow V \otimes W$$

and $g''(w \otimes v) = v \otimes w$.
If we pay attention at how the maps f'' and g'' work we have

$$\begin{aligned} f'' \circ g'' &= \mathbb{I}_{W \otimes V} \\ g'' \circ f'' &= \mathbb{I}_{V \otimes W} \end{aligned}$$

so that $W \otimes V$ and $V \otimes W$ are isomorphic.

□

Proposition 6.3 (Isomorphism of $\mathbb{F} \otimes U$ onto U)
Let us consider $\mathbb{F}$ as a 1-dimensional vector space over $\mathbb{F}$. There exists only one isomorphism of $\mathbb{F} \otimes U$ onto U which sends $\rho \otimes u$ into ρu, $\forall \rho \in \mathbb{F}$ and $\forall u \in U$. The same holds for $U \otimes \mathbb{F}$ and U.

Proposition 6.4 (Isomorphism of $(U \otimes V) \otimes W$ onto $U \otimes (V \otimes W)$)
There exists only one isomorphism of $(U \otimes V) \otimes W$ onto $U \otimes (V \otimes W)$ that sends $(u \otimes v) \otimes w$ into $u \otimes (v \otimes w)$, $\forall u \in U$, $\forall v \in V$ and $\forall w \in W$.

We add now some additional observations.

1. The above property implies that it is meaningful to write $U \otimes V \otimes W$ without brackets.
2. By generalizing proposition (6.1) starting from k-vector spaces $U_1, \ldots, U_k$ we can define $U_1 \otimes \ldots \otimes U_k$.
3. By generalizing proposition (6.2) to the case of the k-fold tensor product[3] $\forall \pi \in S_k$ there exists only one isomorphism of $U_1 \otimes \ldots \otimes U_k$ onto $U_{\pi(1)} \otimes \ldots \otimes U_{\pi(k)}$ that sends $u_1 \otimes \ldots \otimes u_k$ into $u_{\pi(1)} \otimes \ldots \otimes u_{\pi(k)}$.

[3] S_k is the permutation group of k elements.

4. Without proof we are also going to state the following results:

Proposition 6.5 (Tensor product of functions)
Given vector spaces U_j , V_j*,* $j = 1, 2$*, and given maps*

$$f_j : U_j \longrightarrow V_j \quad , \quad j = 1, 2 \quad ,$$

there exists only one map f*,*

$$f : U_1 \otimes U_2 \longrightarrow V_1 \otimes V_2$$

such that $f(u_1 \otimes u_2) = f(u_1) \otimes f(u_2)$ *for all* $u_1 \in U_1$ *and* $u_2 \in U_2$*. By definition we will write*

$$f \stackrel{\text{def.}}{=} f_1 \otimes f_2 \quad .$$